Algebraic structures as seen on the Weyl Algebra

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Introduction

Recently there has been a growing number of interactions between noncommutative algebra and theoretical physics, often via noncommutative geometry. The algebras appearing are usually described by generators and relations what does not provide too much information about their structural properties. A well-known example of this type is the first Weyl algebra, also known as the basic algebra of quantum mechanics because of its direct link to the Heisenberg uncertainty relation. The Weyl algebra is an interesting example also from the algebraic point of view, it is more complicated than a matrix ring, yet it enjoys many structural properties. The idea behind these lecture notes is to develop some algebraic structure theory based upon properties observed on the Weyl algebra.

One of the main methods is the use of filtrations and associated graded rings which is the topic of Chapter 3. The contents deals with : the relation with Lie algebras via the Heisenberg Lie algebra, finitelness conditions on rings and modules, localizations and rings of fractions, the relatioin to rings of differential operators, homological dimensions and the Gelfand-Kirillov dimension, module theory and holonomic modules, simple Noetherian algebras and semisimple rings and modules, etc... . The first three chapters are treated in a more tutorial way; to each of these chapters there is a set of "right or wrong" exercises which can be used as a self-evaluation test by the reader. The second part of the notes is perhaps less tutorial; it should be viewed as a course on the level of the master education (fourth or fifth year of education).

Each Chapter contains some fundamental results; amongst the final goals : the Weyl algebra as a ring of differential operators in Chapter 1, Engel's theorem, Lie's theorem, Cartan's criterion, and Weyl's theorem in Chapter 2; the Bernstein filtration on the Weyl algebra, the control of filtered objects by associated graded objects in Chapter 3; the lifting of Noetherian properties from the graded to the filtered rings and the relation between finitely generated associated graded modules and good filtrations, Maschke's theorem, the Wedderburn, Artin-theorem and Wedderburn's theorem in Chapter 4; Ore's theorem, Goldie's theorem for Noetherian rings in Chapter 5; the GK-dim of an Ore extension, the ideal invariance of GK-dim for a Noetherian ring, exactness of GK-dim of filtered algebras with finitely generated left Noetherian associated graded algebra and the Bernstein inequality in Chapter 6; the calculation of global dimension for filtered rings and the Weyl algebra in Chapter 7.

The material is self-contained; for Chapter 7 some knowledge concerning projective and injective modules is assumed and some basic constructions of homological algebra like taking the cohomology of some (exact) sequences are used without much explication, so this chapter may be treated as a more advanced part, requiring some preknowledge or additional reading.

Acknowledgement

I am grateful towards Raf Bocklandt for composing the "Right or wrong" self-evaluation tests; I also thank all students of the Associative Algebra course for their interest in the subject.

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Contents

Chapter 1

Algebras given by Generators and Relations

1.1 The Free Algebra

We consider a field K and a K-algebra A, that is an associative ring with unit element 1, containing K in its centre $Z(A) = \{x \in A, xa = ax$ for all $a \in A\}$. This entails that A is a K-vector space with a K-bilinear multiplication. A morphism between K-algebras A and B, say $f : A \to B$ is a K linear map respecting multiplication : $f(ab) = f(a)f(b)$ for all $a, b \in A$ and $f(1) = 1$. A subset $\{a_i, i \in J\}$ of A is called a **set of generators** for A/K if every element $x \in A$ is a finite K-linear combination of products of some of the $a_i, i \in J$. If a finite set of generators exists for A then we say that A is a **finitely generated K-algebra**; we denote this by : $A = K < a_1, \ldots, a_n >$. If A is moreover commutative then we write $A = K[a_1, \ldots, a_n]$. An algebra over K is **finite dimensional** if it is finite dimensional as a K-vector space. Every finite dimensional algebra is finitely generated but the converse is not true as can be seen on the polynomial algebra $K[X]$. An element x in a K-algebra A is :

i) A zero-divisor, if there is a $y \in A$ such that $xy = 0$ or $yx = 0$.

ii) An idempotent, if $x^2 = x$

iii) A nilpotent, if $x \neq 0$ but $x^n = 0$ for some $n \in \mathbb{N}$.

iv) A unit, if there is a $y \in A$ such that $xy = yx = 1$.

For a K-algebra A, a **module** will be understood to be a left module; an A-bimodule is a module that is also a right module, M say, such that $\lambda m = m\lambda$ for every $m \in M, \lambda \in A$.

Now consider a set S. The elements of S are considered as symbols or letters of an alphabet. Write $W(S)$ for the set of words in the letters of S, this is the set of all finite ordered sequences of elements of S. Put $K < S >$ equal to the vectorspaces over K with basis $W(S)$, that is $K < S >$ is consisting of all finite K-linear combinations of words in the alphabet S. We can define a multiplication on $K < S >$ by defining the product of words $s_1 \ldots s_n$ and $t_1 \ldots t_m$ as the concatenation $s_1 \ldots s_n.t_1 \ldots t_m$ and extending this by K-bilinearity to K-linear combinations of words. The empty word is the unit for this multiplication and we can embed K in $K < S >$

as the K-multiples of the empty word. Thus $K < S >$ is a K-algebra and we call it the **free K-algebra on** S. To avoid confusion we shall denote generators of a free algebra by capital letters ! Free algebras enjoy a certain generic property expressed in the following.

1.1.1 Lemma

Every K-algebra A is the epimorphic image of some free algebra.

Proof Observe that every K-algebra A has a set of generators (for example A itself); therefore we may select a set of generators $S = \{a_i, i \in J\}$ for the K-algebra A. Let $\{X_i, i \in J\}$ correspond bijectively to this and look at the free algebra $K < X_i, i \in J >$. The K-linear map defined by $\Psi : X_{\alpha_1} \dots X_{\alpha_k} \mapsto a_{\alpha_1} \dots a_{\alpha_k}$ is obviously surjective. It is also easy to veryfy that Ψ is an algebra morphism. $\square$

1.1.2 Corollary

If A is a finitely generated K-algebra then there is an epimorphism

$$K < X_1, \dots, X_n > \longrightarrow\!\!\!\!\rightarrow A$$

for some finite set $\{X_1, \dots, X_n\}$.

A subset L of A such that $AL \subset L$ is a left ideal of A, symmetrically $R \subset A$ such that $RA \subset R$ is a right ideal of A. By an ideal I we will always mean a two-sided ideal, that is : $AIA \subset I$. For an ideal I of A we have the quotient ring A/I and a canonical epimorphism $A \to A/I$. For an algebra morphism $f : A \to B$ we have $\mathrm{Ker}(f) = \{a \in A.f(a) = 0\}$ which is an ideal of A and $\mathrm{Im}(f) = \{f(a), a \in A\}$ which is a K-subalgebra of B such that $A/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$. For any subset $\mathcal{R} \subset A$ we let $(\mathcal{R})$ stand for the ideal generated by $\mathcal{R}$, that is :

$$(\mathcal{R}) = \left\{ \sum_j l_j x_j r_j, x_j \in \mathcal{R}; l_j, r_j \in A \right\}$$

In view of Lemma 1.1.1. every algebra A may be viewed as $K < \mathcal{X} > /(\mathcal{R})$, where the set $\mathcal{X}$ corresponds to a set of K-generators for A and $(\mathcal{R})$ is an ideal called an **ideal of relations** for A. Of course the presentation of A as $K < \mathcal{X} > /(\mathcal{R})$ is not unique.

1.1.3 Example

The kernel of the canonical morphism $\pi : K < X, Y > \to K[X, Y]$ is $(XY - YX)$, thus we have :

$$K[X, Y] = K < X, Y > /(XY - YX)$$

Proof Put $I = (XY - YX), K < a, b >= K < X, Y > /I$ with $a = X \bmod I, b = Y \bmod I$. Then $K < a, b >$ is commutative and π factorizes via $\overline{\pi} : K[a, b] \to k[X, Y]$ with $\overline{\pi}(a) = X, \overline{\pi}(b) = Y$. Any $f \neq 0$ in $\mathrm{Ker}(\overline{\pi})$ would lead to a nontrivial $f(X, Y) = 0$ contradicting the independence of X and Y as variables in the polynomial algebra $K[X, Y]$, hence $\mathrm{Ker}(\overline{\pi}) = 0$ or $\overline{\pi}$ is an isomorphism. $\square$

1.1.4 Examples

Direct sum, free product, tensor product.

i) *Direct Sum*

The direct sum of algebras A and B is the Cartesian product $A \times B$ with the componentwise sum and product. The linear map $A \to A \oplus B : a \mapsto (a, 0)$ is not an algebra morphism as $1 \mapsto (1, 0) \neq I = (1, 1)$ in $A \oplus B$. The projections $\pi_A : A \oplus B \to A$ and $\pi_B : A \oplus B \to B$ are algebra morphisms.

The direct sum has the following universal property : if $\phi_A : C \to A$ and $\phi_B : C \to B$ are algebra morphisms then there is a unique algebra morphism $\phi : C \to A \oplus B$, $n \mapsto (\phi_A(x), \phi_B(x))$, such that $\phi_A = \pi_A \phi$ and $\phi_B = \pi_B \phi$. Now if $A = K < \mathcal{X} > /(\mathcal{R})$ and $B = K < \mathcal{Y} > /(\mathcal{S})$, then $A \oplus B = K < \mathcal{X} \cup \mathcal{Y} \cup \{Z\} > /(Z^2 = Z, Z\mathcal{X} = \mathcal{X}Z = \mathcal{X}, Z\mathcal{Y} = \mathcal{Y}Z = 0, (1 - Z)\mathcal{S}, Z\mathcal{R})$.

Here $Z\mathcal{R}$ stands for the set of relations from $\mathcal{R}$ multiplied by Z. The relations express that Z stands for the element $(1, 0)$. Verify this.

ii) *The Free Product*

We denote by $A*B$ the free product defined by generators and relations (A and B as before) by : $A*B = K < \mathcal{X} \cup \mathcal{Y} > /(\mathcal{R} \cup S)$. There are canonical injections $i_A : A \to A * B, i_B : B \to A * B$, defined by $i_A(X\mathrm{mod}(\mathcal{R})) = \mathcal{X}\mathrm{mod}(\mathcal{R} \cup S), i_B(\mathcal{Y}\mathrm{mod}(S)) = \mathcal{Y}\mathrm{mod}(\mathcal{R} \cup S)$. Now there need not exist projections $A * B \to A$ or $A * B \to B$ (for example when B is a matrix algebra there is no projection $A * B \to A$).

The free product has the following universal property : if $\phi_A : A \to C$ and $\phi_B : B \to C$ are algebra morphisms, then there is a unique algebra morphism $\phi : A * B \to C$ such that $\phi_A = \phi i_A, \phi_B = \phi i_B$.

iii) *The Tensor Product*

The tensor product of A and B (algebras as before), denoted by $A \otimes B$ (meaning $A \otimes_K B$), is described by generators and relations as follows :$A \otimes B = K < \mathcal{X} \cup \mathcal{Y} > /(\{X_i Y_j = Y_j X_i\} \cup \mathcal{R} \cup \mathcal{S})$. Observe that when $\{a_i, i \in I\}$ is a K-basis for A and $\{b_j, j \in J\}$ a K-basis for B then $\{a_i b_j, i \in I, j \in J\}$ is a K-basis for $A \otimes B$. The tensor product is distributive with respect to the direct sum : $A \otimes (B \oplus C) \cong A \otimes B \oplus A \otimes C$, the isomorphism is given by $\phi : A \otimes (B \oplus C) \to A \otimes B \oplus A \otimes C$, $a(b, c) \mapsto (ab, ac)$.

1.2 The Weyl Algebras

In quantum mechanics the Heisenberg uncertainty principle expresses that place and velocity of a particle cannot be simultaneously measured. Measuring in physics means diagonalizing operators (matrices) and noncommutative operators cannot be diagonalized simultaneously (by the same basis-transformation). If x is the place vector of a particle and p its impulse then (writing them as operators): $px - xp = -i\hbar$, where $\hbar$ stands for the Planck constant and $i \in \mathbb{C}$.

Looking at the $\mathbb{C}$-algebra generated by p and x it is clear that we may introduce $y = \frac{ip}{h}$ and look at the algebra : $\mathbb{A}_1(\mathbb{C}) = \mathbb{C} < X, Y > /(YX - XY - 1)$. Let $\phi : \mathbb{C} < X, Y > \to \mathbb{A}_1(\mathbb{C}), X \mapsto x = X \text{mod}(YX - XY - 1)$, be the canonical epimorphism. The $\mathbb{C}$-algebra $\mathbb{A}_1(\mathbb{C})$ is called the **first Weyl algebra**, often referred to as the basic algebra of quantum mechanics. We may replace $\mathbb{C}$ by any field K and consider the first Weyl K-algebra : $K < X, Y > /(YX - XY - 1) = K < x, y >$.

We can also define $\mathbb{A}_n(K) = \mathbb{A}_1(K) \otimes \ldots \otimes \mathbb{A}_n(K)$ with n factors in the tensor product. By iteration of Example 1.1.4. iii. we can describe $\mathbb{A}_n(K)$ by generators and relations as : $\mathbb{A}_n(K) = K < X_1, \ldots, X_n, Y_1, \ldots, Y_n > /(X_iY_j - Y_jX_i$ for $i \neq j, X_iX_j - X_jX_i, Y_iY_j - Y_jY_i$ for i and j, $Y_iX_i - X_iY_i - 1$ for $i = 1, \ldots n)$.

We call $A_n(K)$ the n^{th} Weyl R-algebra. It is usual to write words over $\{x, y\}$ in $\mathbb{A}_1(K)$ with powers of x before powers of y; for example $yx^2 = (yx)x = (1 + xy)x = x + x(yx) = x + x(1 + xy) = 2x + x^2y$. In fact one can prove that every element of $\mathbb{A}_1(K)$ can in a unique way be written as a combination of such words, in other words :

1.2.1 Theorem

The linear map $\imath : K[X, Y] \to \mathbb{A}_1(K)$, $X^iY^j \mapsto x^iy^j$, is a bijective map.

We postpone the proof of this theorem to Chapter 3. Observe that $\imath$ is not a K-algebra morphism, indeed :

$$\imath(YX) = \imath(XY) = xy = yx - 1 = \imath(Y)\imath(X) - 1 \neq \imath(Y)\imath(X)$$

However, $\imath|K[X]$ and $\imath|K[Y]$ are algebra morphisms.

1.2.2 Corollary

The element $x \in \mathbb{A}_1(K)$ is transcendental over K (similar for $y \in \mathbb{A}_1(K)$).

Proof If $f(x) = 0$ for some $f \in K[T]$ then $\imath f(X) = 0$ entails that $f(X) = 0$ hence $f = 0$ because X is transcendental over K. $\square$

1.2.3 Property

For $p(x) \in K[x] \subset \mathbb{A}_1(K)$ we have that $yp(x) - p(x)y = \frac{\partial p}{\partial x}$.

Proof Observe

$$\begin{aligned} yx^2 - x^2y &= (yx)x - x^2y = (xy + 1)x - x^2y \\ &= x(yx) + x - x^2y \\ &= x(1 + xy) + x - x^2y \\ &= x + x^2y + x - x^2y = 2x \end{aligned}$$

Similarly one calculates : $yx^n - x^ny = nx^{n-1}$. Using distributivity $y(z_1 + z_2) - (z_1 + z_2)y = (yz_1 - z_1y) + (yz_2 - z_2y)$, and regrouping terms one obtains : $yp(x) - p(x)y = \frac{\partial p}{\partial x}$. $\square$

On the polynomial ring $K[X]$ we consider the K-linear maps $\xi : K[X] \to K[X], p(X) \mapsto Xp(X)$, $\eta : K[X] \to K[X], p(X) \mapsto \frac{\partial p(X)}{\partial X}$. Consider the K-algebra generated by ξ and η in $\mathrm{End}_K K[X]$.

1.2.4 Proposition

We have $\mathbb{A}_1(K) \cong K < \xi, \eta >$ for K a field of characteristic zero.

Proof Let $\eta\xi - \xi\eta$ act on $p \in K[X]$, then we obtain : $(\eta\xi-\xi\eta)(p(x) = (\frac{\partial}{\partial X}.X - X.\frac{\partial}{\partial X})(p(X)) = \frac{\partial}{\partial X}(Xp(X)) - Xp'(X) = p(X) + X(p'(X)) - X(p'(X)) = p(X)$. Hence the kernel of $\pi : K < X, Y > \to K < \xi, \eta >, X \mapsto \xi, Y \mapsto \eta$, contains the ideal $(YX - XY - 1)$. This means that the canonical map $\widetilde{\pi} : \mathbb{A}_1(K) \to K < \xi, \eta >$, $x \mapsto \xi, y \mapsto \eta$, is well-defined and surjective. Take $p \in \mathrm{Ker}\widetilde{\pi}$. We may write $p = p_0(x) + p_1(x)y + \ldots + p_n(x)y^n$ and from $\widetilde{\pi}(p) = 0$ it follows that $\widetilde{\pi}(p)$ acting or X^i, for every i, is zero. The latter means : $0 = \widetilde{\pi}(p)X^i = \sum_{0 \le j \le i} p_j(X)\frac{i!}{(i-j)!}(-1)^j X^{i-j}$ (where $i! \neq 0$ because $\mathrm{ch}(K) = 0$).

Starting with $i = 0$ we may conclude from the foregoing that $p_i = 0$ for all i, thus $p = 0$. □

Look at a commutative K-algebra D. An additive map $\delta : D \to D$ is said to be a **derivation** if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in D$, if δ is moreover k-linear for some subfield k of K then δ is said to be a k**-derivation**. The set of k-derivations $\delta : D \to D$, say $\mathrm{Der}_k D$, is a K-vector space. It is easily checked that for derivations δ_1, δ_2 also $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$ is a derivation. Let $\mathcal{D}$ be a D-submodule of $\mathrm{Der}_k D$ and write $D < \mathcal{D} >$ for the k-algebra generated by D and $\mathcal{D}$ within $\mathrm{End}_k D$. In case $k = K, \mathcal{D} = \mathrm{Der}_K D$ we obtain $\Delta(D) = D < \mathrm{Der}_K D >$, called the K**-derivation ring of** D or the **ring of differential operators on** D. In case $D = K[X_1, \ldots, X_n]$ then $\mathrm{Der}_K D$ is $D\frac{\partial}{\partial X_1} \oplus \ldots \oplus D\frac{\partial}{\partial X_n}$ and the derivation ring $\Delta(K[X_1, \ldots, X_n]) = \Delta(K[X_1]) \otimes_K \ldots \otimes_K \Delta(K[X_n])$. In view of Proposition 1.2.4. we have the following.

1.2.5 Corollary

$\mathbb{A}_1(K)$, K a field with $\mathrm{ch}(K) = 0$, is isomorphic to $\Delta(K[X])$ and $\mathbb{A}_n(K) \cong \Delta(K[X_1, \ldots, X_n])$.

1.3 Exercises and Examples

1.3.1 The algebra of $n \times n$-matrices $M_n(K)$

a. Describe $M_n(K)$ by generators and relations for $M_n(K)$. How many generators are minimally necessary ?

b. Describe the left, right, two-sided ideals of $M_n(K)$

c. We can embed $\mathbb{C}$ in $M_n(\mathbb{R})$; give an example of this. Why is $M_n(\mathbb{R})$ not a $\mathbb{C}$-algebra ?

1.3.2 The group algebra KG

For a given group G and field K we may define the group algebra KG as the vector space over K with basis G with product defined as the bilinear extension of the product of basis elements in G.

a. For finite abelian proups G describe KG by generators and relations.

b. Show that KG is never a free algebra nor a polynomial ring.

c. Establish that all rational functions on $\mathbb{C}$ that are continuous in $\mathbb{C}^* = \mathbb{C} - \{0\}$ form a group algebra.

d. Do there exist group algebras that are fields ?

e. Show that the tensor product of K-group algebras equals the group algebra over K of the product of the groups, $K(G \times H) \cong KG \otimes_K KH$.

1.3.3 The exterior algebra or Grassmann algebra

Consider n variables $X_1, \ldots, X_n$; the exterior algebra is presented as $\Lambda_n = K < X_1, \ldots, X_n > / \{X_i X_j + X_j X_i, 1 \leq i, j \leq n\}$.

a. Establish that Λ_n is finite dimensional if $\mathrm{ch}(K) \neq 2$ and give the dimension. What happens if $\mathrm{ch}(K) = 2$?

b. What are units, nilpotents, zero-divisors in Λ_n ?

c. Describe the ideals in Λ_2.

1.3.4 Path algebras of quivers

A quiver Q is an oriented graph existing of a set of vertices V and a set of arrows A. We consider maps s and t from $A \to V$, s associates the starting point to an arrow, t associates the end point to an arrow. Observe that we allow multiple arrows between vertices as well as loops (arrows with same start and end points). The path algebra $\mathbb{C}Q$ of a quiver Q is defined as $\mathbb{C}Q = \mathbb{C} < V \cup A > /m$, where m is the ideal generated by the following relations :

- $v^2 = v$ for every $v \in V$
- $vw = 0$ for $v \neq w$ in V
- $as(a) = a = t(a)a$ for every $a \in A$

Observe that the product of arrows is zero when the arrows do not follow each other from right to left, that is :

$$ⓥ \leftarrow ⓦ . ⓤ \leftarrow ⓩ = 0$$

if and only if $w \neq u$!

a. Give a $\mathbb{C}$-basis for $\mathbb{C}Q$ as a vector space.

b. For which quivers is $\mathbb{C}Q$ commutative ?

c. When is $\mathbb{C}Q$ finite dimensional ?

d. Prove that the direct sum of path algebras is again a path algebra.

e. Look at the subalgebra of $M_n(K)$ consisting of the upper-triangular matrices; establish that this is a path-algebra.

1.3.5 The quaternion algebra

Define the $\mathbb{R}$-algebra $\mathbb{H}$ by :

$$\mathbb{H} = \mathbb{R} < i, j > /(ij + ji, i^2 + 1, j^2 + 1)$$

a. Show $\dim_{\mathbb{R}} \mathbb{H} = 4$.

b. Show that $\mathbb{H}$ is a skewfield (each nonzero element has an inverse in $\mathbb{H}$).

c. Prove that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

d. Prove that there are no finite dimensional skewfields over $\mathbb{C}$ and that $\mathbb{H}$ is the unique finite dimensional skewfield over $\mathbb{R}$.

1.3.6 Clifford algebras

Given an n-dimensional K-vector space V with a metric $g : V \times V \to K$ (a nondegenerate quadratic form). Let $\{e_1, \ldots, e_n\}$ be a fixed K-basis of V. We define : $C(V, g) = K < e_1, \ldots, e_n > /(e_i e_j + e_j e_i + 2g(e_i, e_j), 1 \leq i, j \leq n)$

a. Show that the above definition is independent of the chosen basis $\{e_1, \ldots, e_n\}$. How many non-isomorphic Clifford algebras of dimension n exist over $\mathbb{C}$?

b. Establish that $C(V, g)$ is finite dimensional, what is the dimension ?

c. Show that $C(\mathbb{C}^2, g) \cong M_2(\mathbb{C})$.

d. Prove that $C(\mathbb{R}, 1) \cong \mathbb{C}$ but $C(\mathbb{R}, -1) \cong \mathbb{R} \oplus \mathbb{R}$, $C(\mathbb{R}^2, 1) \cong \mathbb{H}, C(\mathbb{R}^3, 1) = \mathbb{H} \oplus \mathbb{H}$

1.4 Wrong or Right

Are the following statements right or wrong ? If the statement is right prove it, if it is wrong, give a counter example. Unless otherwise mentioned, the statements deal with algebras over an arbitrary field K.

1. An algebra generated by one element is commutative.

2. Every finite dimensional commutative algebra over $\mathbb{C}$ can be generated by 1 element.

3. If an algebra is generated by n elements then every subalgebra can be generated by (at most) n elements.

4. If an algebra is generated by n elements then every surjective image can be generated by n elements.

5. A free algebra $K < \mathcal{X} >$ with $|\mathcal{X}| = n$ cannot be generated by less than n elements.

6. The units of a free algebra are the elements of $K - \{0\}$.

7. The field of fractions $K(X)$ of $K[X]$ is a finitely generated K-algebra.

8. For any algebra A there exists a K-algebra morphism $\phi : A \to F$, where F is a free K-algebra.

9. Every algebra that is a surjective image of an algebra having nontrivial idempotents has nontrivial idempotents.

10. A commutative algebra with zero-divisors must have nontrivial ideals.

11. The free product of two free algebras is a free algebra.

12. The free product of commutative algebras is commutative.

13. The direct sum of commutative algebras is commutative.

14. Every algebra that is a two-dimensional as a vector space is a commutative algebra.

15. The free product of two finite dimensional algebras is again finite dimensional.

16. The tensor product of algebras without zero divisors is again an algebra without zero divisors.

17. If a direct sum of algebras contains nilpotents then each term contains (nontrivial) idempotents.

18. A twodimensional algebra which is not direct sum of two algebras contains nontrivial nilpotent elements.

19. A commutative algebra containing a nontrivial idempotent is a direct sum of two algebras.

20. If $\phi_A : C \to A$ is a surjective morphism of algebras and $\phi_B : C \to B$ a morphism then $\phi_{A \oplus B} : C \to A \oplus B$ is surjective.

21. Nilpotent elements plus zero form an ideal in any algebra.

22. The path algebra of a quiver is a subalgebra of a matrix algebra if there is at most one arrow between vertices.

23. The matrix algebra is a path algebra of a quiver.

24. If a free finitely generated algebra A has a surjective morphism $A \to A$ then this morphism is an isomorphism.
25. A finitely generated algebra with a set of generators consisting of invertible elements is a field.
26. If $(a_1, \ldots, a_n)$ generates A and M is an invertible $n \times n$-matrix over A then $\{\sum_j M_{ij} a_j, i = 1, \ldots, n\}$ is also a set of generators ?
27. The automorphism group of the free algebra $\mathbb{F}_2[X]$ has exactly six elements.
28. The number of elements in a algebra is either infinite or the power of a prime?
29. An algebra with no nontrivial subalgebras is commutative ?
30. An algebra with no nontrivial subalgebras is a field.
31. An algebra generated by two commuting nilpotent elements is finite dimensional.
32. A free finitely generated algebra contains finitely many commutative subalgebras.
33. In an algebra over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, taking the p-th power is an algebra morphism.
34. Let X be a set, A an algebra. There is a bijection between the set of maps $X \to A$ and morphisms $K < X > \to A$.
35. There are no morphisms from the Weyl-algebra to a matrix algebra $M_n(\mathbb{C})$.
36. Every nontrivial finite dimensional $\mathbb{C}$-algebra must contain nontrivial zero divisors.
37. Every injective endomorphism of a finite dimensional algebra is necessarily an automorphism.
38. The endomorphisms of an algebra form an algebra with composition as multiplication.
39. Every finite dimensional algebra can be embedded in a matrix algebra.
40. The tensor product of matrix algebras is a matrix algebra.
41. Every morphism from the quaternions to another algebra is an embedding.
42. Commutative subalgebras of the Grasmann algebra over $\mathbb{C}$ are generated by one element.
43. The Clifford algebra has no nilpotent (nontrivial) elements.
44. The Grassmann algebra Λ_n may be mapped surjectively onto Λ_{n-1}.
45. The Grasmann algebra Λ_{n-1} can be embedded into Λ_n.
46. The Clifford algebra $C(\mathbb{C}^n, 1)$ can be mapped surjectively onto $C(\mathbb{C}^{n-1}, 1)$.
47. A path-algebra without zero divisors is a free algebra.

48. A path algebra without nilpotent elements is a free algebra.

49. A path algebra which can be mapped surjectively onto $\mathbb{C}[X]$ necessarily has a loop.

50. The free product of two path algebras is a path algebra.

Chapter 2

Lie Algebras and Derivations

2.1 Derivations and their Invariants

Let A be a K-algebra and M a K-vector space. We say that M is a **A-bimodule** over K if it is a left and right A-module, i.e. $(am)b = a(mb)$ for $a, b \in A$ and $m \in M$, such that $\lambda m = m\lambda$ for every $\lambda \in K$. An additive map $\delta : A \to M$ is called a **derivation** if $\delta(ab) = a\delta(b) + \delta(a)b$, for every $a, b \in A$; it is said to be a **K-derivation** if δ is K-linear. Observe that δ id K-linear if and only if $\delta|K = 0$. Considering A as an A-bimodule with respect to the ring multiplication we obtain $\mathrm{Der}(A)$, the set of derivations $\delta : A \to A$, and also $\mathrm{Der}_K(A)$, the set of K-derivations from A to A. A derivation $\delta : A \to M$ is an **inner** ($K-$) **derivation** if there is an $m \in M$ such that $\delta(a) = ma - am$ for all $a \in A$; an inner derivation is K-linear by definition. We write $\mathrm{In}_K(A)$ for the set of inner derivations.

2.1.1 Definition

A K-vector space g is a Lie algebra if there exists a bilinear map $[-,-] : g \times g \to g$, such that $[-,-]$ is antisymmetric and satisfies the Jacobi identity : $\forall a, b, c \in g$ we have :

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

The operation $[-,-]$ is called a **Lie Bracket**.

2.1.2 Properties

1. For $\delta_1, \delta_2 \in \mathrm{Der}(A)$ define $[\delta_1, \delta_2]$ by $\delta_1\delta_2 - \delta_2\delta_1$, then $\mathrm{Der}(A)$ as well as $\mathrm{Der}_K(A)$ are Lie algebras with this bracket. By δ_x for $x \in A$ we mean the inner derivation defined by $x \in A$, then we have :$[\delta_x, \delta_y] = \delta_{[x,y]}$, and thus $\mathrm{In}_K(A)$ is a Lie subalgebra of $\mathrm{Der}_K(A)$.

2. $\mathrm{In}_K(A)$ is a Lie ideal of $\mathrm{Der}_K(A)$, that is for every $\delta \in \mathrm{Der}_K(A)$ and every $\delta_x \in \mathrm{In}_K(A)$ we have that $[\delta, \delta_x] \in \mathrm{In}_K(A)$.

Proof Easy verification (exercise) ! □

The **invariants** for $\delta \in \mathrm{Der}(A)$ are given by $A^\delta = \{a \in A, \delta(a) = 0\}$. For $X \subset \mathrm{Der}(A)$, $A^X = \{a \in A, \delta(a) = 0 \text{ for all } \delta \in X\}$, if $X \subset \mathrm{Der}_K(A)$ then A^X is a K-vector space.

For $x \in g$ we have $ad\ x : g \to g, y \mapsto [x, y]$, and this is a K-linear map.

Now let us consider a K-algebra A with $\mathrm{ch}(K) = p > 0$ and define for $x, y \in A$, $[x, y] = xy - yx$. Also we may again consider $ad\ x : A \to A, y \mapsto [x, y]$. Clearly A with this $[-, -]$ is a Lie algebra but it also satisfies the following :

i) $[x^p, y] = [x, \ldots, [x, y]] \ldots]$, p-fold bracket.

ii) $ad[x, y] = [ad\ x, ad\ y] = adx\ ady - ady\ adx$.

iii) $ad\ x^p = (adx)^p$ see i.

Any additive subgroup B of A such that $[b, b'] \in B$ for $b, b' \in B$, is called a **Lie ring in** A. For a subfield k of K, $\mathrm{Der}_k(K)$ is a Lie ring in $\mathrm{End}_k(K)$ and more generally : $\mathrm{Der}(K)$ is a Lie ring of $\mathrm{End}_+(K) = E$, the ring of additive endomorphisms of K with composition for the multplication. Of course $\mathrm{Der}_k(K) = \mathrm{Der}(K) \cap \mathrm{End}_k(K)$. For $S, T \in \mathrm{Der}(K)$ we have $[S, T] \in \mathrm{Der}(K)$ and also $S^p \in \mathrm{Der}(K)$ (verify this).

A Lie subring of $\mathrm{Der}(K)$ is a K-vector space $\mathcal{D}$ in $\mathrm{Der}(K)$ such that for $S, T \in \mathcal{D}$ we have that $[S, T] = \mathcal{D}$ and $S^p \in \mathcal{D}$.

The **invariant field** $K^{\mathcal{D}}$ is given as : $K^{\mathcal{D}} = \{y \in K, D(xy) = D(x)y$ for every $D \in \mathcal{D}, x \in K\} = \{y \in K, D(y) = 0$ for all $D \in \mathcal{D}\}$. Since $K^p(p = \mathrm{ch}(K))$ is invariant for any derivation we have that $K^p \subset K^{\mathcal{D}}$ for every K-Lie subring $\mathcal{D}$.

2.1.3 Proposition

Consider a field extension $k \subset K$ with $K^p \subset k, p = \mathrm{ch}(K) > 0$. Then $K^{\mathcal{D}} = k$ for $\mathcal{D} = \mathrm{Der}_k(K)$.

Proof That $k \subset K^{\mathcal{D}}$ is clear. Suppose $x \in K - k$ and let k' be a maximal subfield of K not containing x; this exists because $x \notin k$ and Zorn's lemma. Assume $K \neq k'(x)$, so pick $y \in K - k'(x)$. Since $k'(y) \supsetneqq k'$ it follows that $x \in k'(y)$ and thus we arrive at : $k' \subsetneqq k'(x) \subset k'(y)$. Since $K^p \subset k$ we have $x^p \in k'$ as well as $y^p \in k'$ thus $[k'(x) : k'] = p$ and $[k'(y) : k'] = p$ or $k'(x) = k'(y)$, contradicting $y \notin k'(x)$! Therefore we must have $K = k'(x)$. Now let $T : K \to K$ be the k'-linear map defined by $x^i \mapsto ix^i$ for $i = 0, \ldots, p-1$. Since $x^p \in k'$ we have $T(x^i) = ix^i$ for all $i \in \mathbb{N}$, thus $T \in \mathrm{Der}_k K$ with $T(x) \neq 0$ yielding $x \notin K^{\mathcal{D}}$. Consequently : $K^{\mathcal{D}} = k$. □

Consider a Lie K-subring $\mathcal{D}$ in $\mathrm{Der}(K)$. For $x \in K$, define $\widehat{x} : \mathcal{D} \to K, D \mapsto D(x)$; for $X \subset K$ we let $\widehat{X}$ be the set $\{\widehat{x}, x \in X\}$. By definition we have that $\widehat{K} \subset \mathrm{Hom}_K(\mathcal{D}, K)$. We say that $\widehat{X}$ **separates** $\mathcal{D}$ if $\widehat{x}(T) = \widehat{x}(S)$ for all $\widehat{x} \in \widehat{X}$ entails $T = S$. In particular $\widehat{K}$ separates $\mathcal{D}$. W write $\mathbb{F}_p$ for the prime subfield of K (i.e. the subfield generated by 0 and 1, $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, where $p = \mathrm{ch}(K)$). An $\mathbb{F}_p$**-form** for $\mathcal{D}$ is a Lie $\mathbb{F}_p$-subring of $\mathcal{D}$, say $\mathcal{D}_p$, such that $K\mathcal{D}_p = \mathcal{D}$.

2.1.4 Theorem

With notation as above, assume that $\dim_K \mathcal{D} < \infty$. Then $\mathcal{D}$ has an $\mathbb{F}_p$-form consisting of two-by-two commuting and $\mathbb{F}_p$-diagonizable derivations. Moreover we have that $[K : K^{\mathcal{D}}] = p^{[\mathcal{D}:K]}$.

Proof One easily verifies that since $\widehat{K}$ separates $\mathcal{D}, \widehat{K}$ contains a K-basis for $\mathrm{Hom}_K(\mathcal{D}, K)$, say $\{\widehat{x}_1, \ldots, \widehat{x}_n\}$. Let $T_1, \ldots, T_n$ be the K-basis for D defined by $\widehat{x}_j(T_i) = \delta_{ij} x_j$ (dual basis). We now calculate : for $1 \leq i, j, r \leq n$:

$$\begin{array}{ll} (a) & [T_i, T_j](x_r) = T_i(T_j(x_r)) - T_j(T_i(x_r)) = 0 \\ (b) & T_i^p(x_r) = \delta_{ir} x_r = T_i(x_r) \end{array}$$

Since $\widehat{x}_r$, with $r = 1, \ldots, n$ separate $\mathcal{D}$ we obtain from (a) and (b) that $[T_i, T_j] = 0$ and $T_i^p = T_i$ for $i, j \in \{1, \ldots, n\}$. Hence T_i is a solution for the separable polynomial $X^p - X$ but $X^p - X = \prod_{\lambda \in \mathbb{F}_p}(X - \lambda)$ since $X^p - X$ has p solutions in $\mathbb{F}_p$.

Therefore the T_i have p different eigensvalues and so they can be diagonalized simultaneously over $\mathbb{F}_p$ (observe that the T_i are of course $\mathbb{F}_p$-linear). Put $\mathcal{D}_p = \mathbb{F}_p T_1 \oplus \ldots \oplus \mathbb{F}_p T_n$. It is clear that $\mathcal{D}_p$ is an $\mathbb{F}_p$-form of $\mathcal{D}$ consisting of two-by-two commuting $\mathbb{F}_p$-diagonalizable derivations. Put $\mathcal{R} = \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{D}_p, \mathbb{F}_p), \alpha \in \mathcal{R}, K_\alpha = \{x \in K, T(x) = \alpha(T)x$ for $T \in \mathcal{D}_p\}$. Then we claim that $K_0 = K^{\mathcal{D}}, K = \oplus_{\alpha \in \mathcal{R}} K_\alpha$ and $K_\alpha K_\beta \subset K_{\alpha+\beta}$ for $\alpha, \beta \in \mathcal{R}$ (in other words we claim that K is $\mathcal{R}$-graded for the additive group $\mathcal{R}$).

We continue by induction on $\dim_K D$ and it suffices to give the proof in case $\mathcal{D} = KD, \dim_K \mathcal{D} = 1$. In this case, put $K_i = \{x \in K, D(x) = ix\}$ for $i \in \mathbb{F}_p$. Clearly $K_0 = K^{\mathcal{D}} = K^D$ is a subfield of K and every K_i is a K_0-space. That $K = \oplus_{i \in \mathbb{F}_p} K_i$ follows from the decomposition according to the different eigenvalues of D and as $D^p = D$ these are just all the elements of $\mathbb{F}_p$.

Now from : $D(x_{(i)} x_{(j)}) = x_{(i)} D(x_{(j)}) + D(x_{(i)}) x_{(j)} = j x_{(i)} x_{(j)} + i x_{(i)} x_{(j)} = (i + j) x_{(i)} x_{(j)}$, with $x_{(i)} \in K_i, x_{(j)} \in K_j$, it follows that $K_i K_j \subset K_{i+j}$ for $i, j \in \mathbb{F}_p$. Moreover $K = K^{\mathcal{D}}(x)$ and $K_i = K^{\mathcal{D}} x^i$, $i = 0, \ldots, p-1$, for any $x \in K_1 - \{0\}$ because for $y, z \in K_i$ we have $D(z) = iz, D(y) = iy$ hence $D(zy^{-1}) = 0$, yielding $zy^{-1} \in K^{\mathcal{D}}$ or $K^{\mathcal{D}} z = K^{\mathcal{D}} y$. In a similar way one proves that $K_\beta = K_0 x_\beta$ for any $x_\beta \neq 0$ in K_β in the general case. Note : for the induction one can start with $D = T_1$ and then look at the tower :

$$K \supset K^{T_1} \supset K^{\mathbb{F}_p T_1 \oplus \mathbb{F}_p T_2} \supset \ldots K^{\mathcal{D}_p} = K^{\mathcal{D}}$$

Let $R_0 \subset \mathcal{R}$ be the set $\{\beta \in \mathcal{R}, K_\beta \neq 0\}$. Then from $K_\alpha K_\beta \subset K_{\alpha+\beta}$ and the fact that there are no nonzero zero divisors in K, it follows that $\mathcal{R}_0$ is closed for the sum and thus $\mathcal{R}_0$ is an $\mathbb{F}_p$-subspace of $\mathcal{R}$. Obviously $\mathcal{R}_0$ also separates $\mathcal{D}_p$ by definition, thus $\mathcal{R}_0 = \mathcal{R}$. Consequently $K = \oplus_{\alpha \in \mathcal{R}} K_0 x_\alpha$ and thus we arrive at $[K : K^\infty] = |\mathcal{R}| = p^{[\mathcal{D}_p : \mathbb{F}_p]} = p^{[\mathcal{D}:K]}$. □

2.1.5 Theorem (N. Jacobson)

With notation as above, if $\dim_K \mathcal{D} < \infty$ then $\mathcal{D} = \mathrm{Der}_{K^{\mathcal{D}}}(K)$.

Proof That $\mathcal{D} \subset \mathrm{Der}_{K^{\mathcal{D}}}(K)$ is obvious. The foregoing thorem yields :

$$p^{[\mathcal{D}:K]} \underset{2.1.4}{=} [K : K^{\mathcal{D}}] \underset{2.1.3.}{=} [K : K^{\mathrm{Der}_{K^{\mathcal{D}}}(K)}] \underset{2.1.4.}{=} p^{[\mathrm{Der}_{K^{\mathcal{D}}}(K):K]}$$

Therefore $[D : K] = [\mathrm{Der}_{K^{\mathcal{D}}}(K) : K]$, thus $\mathcal{D} = \mathrm{Der}_{K^{\mathcal{D}}}(K)$. □

Jacobson's theorem yields that $k \mapsto \mathrm{Der}_k K$ is an inclusion reversing bijection between the set of subfields $k \subset K$ with $K^p \subset k$ and $[K : k] < \infty$, and the set of Lie K-subrings $\mathcal{D}$ in $\mathrm{Der}(K)$ which are finite dimensional over K.

This correspondence is called the Jacobson differential correspondence, it may be viewed as a version of the Galois correspondence for purely inseparable fieldextensions of exponent one.

2.2 Lie Algebras and their Enveloping Algebras

On an associative algebra A we can define a Lie bracket $[a, b] = ab - ba$, for a and b in A. For any given Lie algebra g one may ask whether there exists an associated algebra $U(g)$ such that its commutator bracket induces the Lie bracket of g in $U(g)$. We will now construct such an algebra, called the enveloping algebra of g. Choose a basis $e_1, \ldots, e_n$ for g and define structure constants c_{ij}^k by : $[e_i, e_j] = \sum_k c_{ij}^k e_k$. One easily verifies that the structure constants satisfy the following conditions :

a. $c_{ij}^k + c_{ji}^k = 0$

b. $\sum_l (c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0$

We now define the **universal enveloping algebra** $U_K(g)$ by generators and relations :

$$U_K(g) = K < X_1, \ldots, X_n > /(X_i X_j - X_j X_i - \sum_k c_{ij}^k X_k, 1 \leq i, j \leq n)$$

One easily checks that the above definition is independent of the chosen K-basis for g. The term "universal" points at the following universal property for the eveloping algebra. Given an algebra A and Lie algebra homomorphism $\phi : g \to A$, where on A we use the commutative bracket, then there is a unique algebra morphism $\overline{\phi} : U(g) \to A$ such that $\overline{\phi}|g = \phi$, that is there is unique algebra morphism $\overline{\phi}$ making the following triangle commutative :

$$\begin{tikzcd} g \arrow[r, "\phi"] \arrow[d, hook] & A \\ U(g) \arrow[ur, "\overline{\beta}"'] & \end{tikzcd}$$

We have seen that the derivations of an algebra form a Lie algebra and so do the inner derivations; however not every derivation is inner. There is a construction (ring of differential polynomials) making a given derivation of an algebra A inner in the algebra B constructed as an overring of A, $A \subset B$. Let us do this in detail for $A = \mathbb{C}[T]$, the algebra of polynomials in one variable and $\delta = \frac{d}{dT}$, the derivative with respect to T. Since A is commutative δ is not inner in A. Look at $\mathbb{C} < T, Z >$ and the ideal $I = (\frac{dp(T)}{dT} - Zp(T) + p(T)Z), p(T) \in \mathbb{C}[T]$ in it. We put : $\mathbb{C} < T, Z > /I = \mathbb{C}[T][Z, \delta]$. Since $ZT - TZ - 1 \in I$ and we know that $(ZT - TZ - 1)$ contains all relations $\delta p(T) = Zp(T) - p(T)Z$, we obtain $I = (ZT - TZ - 1)$ and thus : $\mathbb{A}_1(\mathbb{C}) \cong \mathbb{C}[T][Z, \delta]$, identifying x with T and y with Z. Thus we see that $\mathbb{A}_1(\mathbb{C})$ is the overring of $\mathbb{C}[x]$ making $\frac{d}{dx}$ into an inner derivarion (induced by y). We now present examples of Lie algebras in a set of exercises (the solutions are given in the chapter solutions of the exercises).

2.2.1 Exercise

The algebra $M_2(\mathbb{C})$ with commutator bracket forms a 4-dimensional Lie algebra. Describe all Lie subalgebras, say which ones are in fact Lie ideals and describe quotient Lie algebras.

2.2.2 Exercise

Describe the $\mathbb{C}$-derivations of the exterior algebras $\Lambda_2(\mathbb{C})$ and $\Lambda_3(\mathbb{C})$, say which ones are interior.

2.2.3 Exercise

Let G be a group, $G \subset M_n(\mathbb{R})$ and let $f : \mathbb{R} \to G$ be a differentiable function such that $f(0) = I$. Define $X_f = \frac{df}{dt}(0)$ and put $g = \{X_f, f(0) = I\}$.

a. Show that g is a Lie algebra. Hint : calculate $\frac{\delta^2}{\delta_s \delta^t} f(t)g(s)f(t)^{-1}|_{s,t=0}$. Why is this in g ?

b. What are the Lie algebras $\mathrm{gl}_n(\mathbb{R}), \mathrm{sl}_n(\mathbb{R})$, $\mathrm{so}_n(\mathbb{R})$ corresponding to the respective subgroups :

$$GL_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}), \det(X) \neq 0\}$$
$$SL_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}), \det(X) = 1\}$$
$$SO_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}), XX^t = I\}$$

What are their dimensions as $\mathbb{R}$ vector spaces ?

c. Establish a Lie algebra isomorphism between $so_3(\mathbb{R})$ and $\mathbb{R}^3, \times$ ($\times$ the vectorial product).

d. Do there exist Lie groups with the same Lie algebra ?

e. Establish that the Lie subalgebra corresponding to a closed normal subgroup is in fact a Lie ideal. The same statements may be formulated over $\mathbb{C}$ instead of $\mathbb{R}$.

2.2.4 Exercise

An important class of complex Lie algebras is consisting of **simple Lie algebras**, these are Lie algebras having no non-trivial Lie ideals.

a. Prove that no complex Lie algebra of dimension 2 is simple.

b. Prove that $\mathrm{sl}_2(\mathbb{C}$ is simple by checking that it has a $\mathbb{C}$-basis K, E, F such that : $[K, E] = 2E, [K, F] = -2F, [E, F] = K$.

c. In $sl_3(\mathbb{C})$ we may embed $\mathrm{sl}_2(\mathbb{C})$ as a Lie subalgebra in two ways. First by restricting to the first (from the left) two columns and two rows (from the top) and secondly by using the last two columns and rows. Using the basis as in the foregoing exercise we obatin E_i, F_i, K_i with $i \in \{1, 2\}$. Now $\mathrm{sl}_3(\mathbb{C})$ has dimension 8 so how can we express the two remaining basis elements for $\mathrm{sl}_3(\mathbb{C})$ in terms of the E_i, F_i, K_i ?

d. Let Q be a quiver with n vertices and no loops such that all arrows between two vertices run in the same direction. The Cartan-matrix for Q is an n by n matrix A having all diagonal entries equal to 2 and $A_{ij} = -1$ if there are arrows from i to j and $A_{ij} = -k$ if k arrows run from j to i. We look at the free algebra $\mathbb{C} < K_i, E_i, F_i, 1 \leq i \leq n >= \mathcal{F}$ and look at the ideal I of $\mathcal{F}$ generated by the relations :

i) $K_iK_j - K_jK_i$

ii) $K_iE_j - E_jK_i - a_{ij}E_j, K_iF_j - F_jK_i + a_{ij}F_j$, where $A = (a_{ij})$.

iii) $E_iF_j - F_jE_i - \delta_{ij}K_i$

iv) $\underbrace{[E_i[E_i \ldots [E_i, E_j] \ldots]}_{a_{ij}-1 \text{ terms}}, \underbrace{[F_i, [F_i[\ldots [F_i, F_j] \ldots]}_{a_{ij}-1 \text{ terms}}$

With these notations the main theorem in Lie theory may be formulated as follows.

2.2.5 Theorem

The quotient algebra $\mathcal{F}/I$ is the enveloping algebra of a finite dimensional simple Lie algebra g if and only if Q is one of the following quivers :

A_n $$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] & \ldots 0 \arrow[r, dash] & 0 \end{tikzcd}$$

B_n $$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] & \ldots 0 \arrow[r, Rightarrow] & 0 \end{tikzcd}$$

C_n $$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, dash] & \ldots 0 & 0 \arrow[l, Rightarrow] \end{tikzcd}$$

D_n $$\begin{tikzcd} & & 0 \arrow[d, dash] & \\ 0 \arrow[r, dash] & 0 \arrow[r, dash] & \ldots 0 \arrow[r, dash] & 0 \end{tikzcd}$$

E_6 $$\begin{tikzcd} & & 0 \arrow[d, dash] & & \\ 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \end{tikzcd}$$

E_7 $$\begin{tikzcd} & & 0 \arrow[d, dash] & & & \\ 0 \arrow[r, dash] & 0 \arrow[r] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \end{tikzcd}$$

E_8 $$\begin{tikzcd} & & 0 \arrow[d, dash] & & & & \\ 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \end{tikzcd}$$

F_4 $$\begin{tikzcd} 0 \arrow[r, dash] & 0 \arrow[r, Rightarrow] & 0 \arrow[r, dash] & 0 \end{tikzcd}$$

G_2 $$\begin{tikzcd} 0 \arrow[r, Rrightarrow] & 0 \end{tikzcd}$$

In that case g is the smallest Lie subalgebra containing the X_i. Conversely, every non-trivial finite dimensional simple Lie algebra is isomorphic to one of these algebras. The quivers in the list are called **Dynkin-diagrams**. Note that orientation on unique arrows need not be indicated because this is encoded in the Cartan matrix.

2.2.6 Exercise

Show that $\mathrm{sl}_n(\mathbb{C})$ corresponds to A_{n-1}.

Hinti : use the E_i, F_i, K_i-generators and extend this to general n.

The diagrams B_n correspond with $\mathrm{so}_{2n+1}(\mathbb{C})$, C_n with $\mathrm{sp}_n(\mathbb{C})$ and D_n with $\mathrm{so}_{2n}(\mathbb{C})$. The others are called **exceptional Lie algebras** and are related to the symmetries of the octonions.

A special example is the **Heisenberg Lie algebra**. It is a three dimensional Lie algebra, say $KX \oplus KY \oplus KZ$ with Lie bracket defined by : $Z = [Y, X], [X, Z] = [Y, Z] = 0$. We denote this Lie algebra by $\mathcal{H}$ and its enveloping algebra by $U_K(\mathcal{H})$ or $U(\mathcal{H})$ for short.

2.2.7 Property

There is an algebra epimorphism $\Psi : U_K(\mathcal{H}) \to \mathbb{A}_1(K), X \mapsto x, Y \mapsto y, Z \mapsto 1$.

Proof The element Z, hence $Z-1$, is central in $U(\mathcal{H})$, hence $U(\mathcal{H})(Z-1)$ is a two-sided ideal of $U(\mathcal{H})$. Clearly $U(\mathcal{H})/U(\mathcal{H})(Z-1)$ is generated by $x = X\mathrm{mod}(Z-1)$, $y = Y\mathrm{mod}(Z-1)$ and $[y, x] = 1$ or $yx - xy = 1$ holds. This is generating the ideal of relations viewing $\mathbb{A}_1(K)$ as an image of $K < X, Y >$ as is easily seen. $\square$

The foregoing property links the Weyl algebra to the Heisenberg Lie algebra, later we shall understand this link via the blow up ring for the operator filtration on the Weyl algebra (see Chapter 3).

2.3 Some Structure Theory for Lie Algebras

We aim to provide the basic theory for some Lie algebras, in particular solvable, nilpotent and semisimple Lie algebras. We refer to the literature for the detail about Lie algebra theory, in particular to [5].

Let us start with some facts about representations of a Lie algebra; we only use very elementary notions from category theory, cf. [Mc] for a detailed study.

For a Lie algebra g we denote by $\mathrm{Rep}(g)$ the category where objects are representations of g, that is Lie algebra maps $q : g \to \mathrm{gl}(V)$ where V is a K-vector space and $\mathrm{gl}(V)$ is like $\mathrm{gl}_n(\mathbb{R})$ in Exercise 2.1.3., the Lie algebra corresponding to $\mathrm{GL}(V)$ of automorphisms of V. Morphisms in $\mathrm{Rep}(g)$ are defined as follows, for representations of g, say $q_1 : g \to \mathrm{gl}(V_1), q_2 : g \to \mathrm{gl}(V_2)$, a linear map $u : V_1 \to V_2$ is a **morphism of representations**, we write $u : q_1 \to q_2$, if for all

$x \in g$ we have : $q_2(x)u = uq_1(x)$, that is the following diagram is commutative for all $x \in g$:

$$\begin{tikzcd} V_1 \arrow[r, "u"] \arrow[d, "q_1(x)"'] & V_2 \arrow[d, "q_2(x)"] \\ V_1 \arrow[r, "u"'] & V_2 \end{tikzcd}$$

A **subrepresentation** of $q : g \to \mathrm{gl}(V)$, is a subspace $W \subset V$ stabilized by $q(x)$, for all $x \in g$: $q(x)W \subset W$ for all $x \in g$. A representation is called **irreducible** if it has no proper subrepresentations. The **direct sum of representations** $q_1 : g \to \mathrm{gl}(V_1)$ and $q_2 : g \to \mathrm{gl}(V_2)$ is defined by $q_1 \oplus q_2 : g \to \mathrm{gl}(V_1 \oplus V_2)$, $(q_1 \oplus q_2)(x) = q_1(x) + q_2(x), x \in g$. A representation is said to be **completely reducible** if any subrepresentations is a direct summand. We define the category of g**-modules** as the category with objects the K-vector spaces V with an operation $g \times V \to V, (x, v) \mapsto x.v$ such that :

LM.1. The operation is bilinear

LM.2. $[x, y].v = x.y.v - y.x.v$ for all $x, y \in g, v \in V$

The morphisms in g-mod are the K-linear maps $\phi : V_1 \to V_2$ such that $\phi(x.v) = x.\phi(v)$ for all $x \in g, v \in V$. A **submodule of a** g**-module** V is a K-subspace $W \subset V$ such that $x.w \in W$ for all $x \in g, w \in W$. A g-module is **irreducible** or **simple** if it has no proper submodules. Direct sums in g-mod as well as completely reducible or semireducible modules are defined as before.

2.3.1 Exercise

Show that the categories $\mathrm{Rep}(g)$ and g-mod are isomorphic.

2.3.2 Example

A Lie algebra g acts on itself via the adjoint representation, ad : $g \mapsto \mathrm{gl}(g)$ where for $x \in g$, $\mathrm{ad}(x)(y) = [x, y]$ for $y \in g$. The g-module g is denoted by ${}_g g$. We see that g is a simple Lie algebra if and only if ${}_g g$ is irreducible and g is semisimple if and only if ${}_g g$ is ompletely reducible.

2.3.3 Lemma (Schur's lemma)

Assume K is algebraically closed. If $q : g \to \mathrm{gl}(V)$ is an irreducible representation then each endomorphism of V commuting with the $q(x), x \in g$, are those of the form αI_V with $\alpha \in K$.

Proof Obviously a morphism $u : V \to V$ as in the statement is a g module morphism, hence $\mathrm{Im}(u)$ is a submodule of V. If $\mathrm{Im}(u) = 0$ then $u = 0$ and everything is evident. If $\mathrm{Im} \neq 0$ then $\mathrm{Im}(u) = V$ as V is irreducible, thus $\mathrm{Ker} u = 0$ and u is an isomorphisn. Since K is algebraically closed u has an eigenvalue $\lambda \in K$. Put $V_\lambda = \{v \in V, u(v) = \lambda v\}$; this is clearly a g-submodule since for all $v \in V_\lambda, x \in g$, $u(x.v) = x.u(v) = \lambda x.v$. Since $V_\lambda \neq 0$ we must have $V_\lambda = V$ and so u is then λI_V.

2.3.4 Proposition

1. If V is a g-module then $V^* = \mathrm{Hom}_K(V, K)$ is a g-module by putting $(x.f)(v) = -f(x.v)$, for all $f \in V^*, v \in V, x \in g$.

2. The above correspondence defines a duality between the category of finite dimensional g-modules and itself.

Proof

1. Bilinearity of the operation is clear.

$$\begin{aligned}([x,y].f)(v) &= -f([x,y].v) = -f(xy.v - yx.v)\\ &= -f(xyv) + f(yx.v) = -(yx.f)(v) + (xy.f)(v)\\ &= (xy.f - yx.f)(v)\end{aligned}$$

2. If $u : V_1 \to V_2$ is a morphism of g-modules, then $u^* : V_1^* \to V_2^*, u^*(f) := f \circ u$, is also a morphism, since

$$\begin{aligned}u^*(x.f)(v_1) &= (x.f)(u(v_1)) = -f(x.u)(v_1)\\ &= -f(u(x.v_1)) = -u^*(f)(x.v_1) = (x.u^*(f))(v_1)\end{aligned}$$

Now the canonical vector space isomorphism

$$\psi : V \to V^{**}, \psi(v)(f) = f(v)$$

is a g-module map, hence it is an isomorphism of g-modules. Indeed, we have $\psi(x,v)(f) = f(x.v)$ and $(x.\psi(v))(f) = -\psi(v)(x.f) = -(x.f)(v) = f(x.v)$.

2.3.5 Proposition

If V and W are g-modules, then $V \otimes_F W$ becomes a g-module via

$$x,(x \otimes w) := x, v \otimes w + v \otimes x.w$$

Proof

$$\begin{aligned}[xy].(v \otimes w) &= [xy].v \otimes w + v \otimes [xy].w\\ &= x.y.v \otimes w - y.x.v \otimes w + v \otimes x.y.w - v \otimes y.x.w\\ &= x.y.v \otimes w + y,v \otimes x.w + x.v \otimes y.w + v \otimes x.y.w -\\ &\quad -y.x.v \otimes w - x.v \otimes y.w - y.v \otimes x.w - v \otimes y.x.w\\ &= x.(y.v \otimes w + v \otimes y.w) - y.(x.v \otimes w + v \otimes x.w)\\ &= x.y.(v \otimes w) - y.x(v \otimes w)\end{aligned}$$

2.3.6 Proposition

Let V and W be finite dimensional g-modules. Then we can turn $\mathrm{Hom}_K(V.W)$ into a g-module via

$$(x.f)(v) = x.f(v) - f(x.v)$$

Proof The linear map

$$\phi : V^* \otimes W \to \mathrm{Hom}_K(V, W), \phi(f \otimes w)(v) = f(v)w$$

is an isomorphism of vector spaces. Indeed, let $\{e_i\}$ and $\{f_j\}$ be bases in V and W, and $\{e_i^*\}$ be the dual basis in V^* (i.e. $v = \sum e_i^*(v)e_i$, $\forall v \in V$). Let $f \in \mathrm{Hom}_K(V, W)$ and $v \in V$. Then $f(v) = (\sum e_i^*(v)f(e_i) = \phi(\sum_i e_i^* \otimes f(e_i))(v)$. Thus ϕ is surjective, and since the two spaces have the same dimension $(\dim_K(V)\dim_K(W))$, ϕ is an isomorpism.

By Proposition 2.3.4. V^* is a g-module, and by Proposition 2.3.5 $V^* \otimes W$ also becomes a g-module. This structure can be carried to $\mathrm{Hom}_K(V, W)$ via ϕ. So let $x \in g$ and $f \in \mathrm{Hom}_K(V, W)$. We have

$$\begin{aligned}
(x.f)(v) = & \\
&= \phi(\sum x(e_i^* \otimes f(e_i))(v) \\
&= \phi(\sum x.e_i^* \otimes f(e_i) + e_i^* \otimes x.f(e_i))(v) \\
&= \sum (x.e_i^*)(v)f(e_i) + \sum e_i^*(v)x.f(e_i) \\
&= \sum (x.e_i^*)(v)f(e_i) + x.f(\sum e_i^*(v)e_i) \\
&= x, f(v) + \sum (x.e_i^*)(v)f(e_i) \\
&= x.f(v) - \sum e_i^*(x.v)f(e_i) \\
&= x.f(v) - f(x.v)
\end{aligned}$$

2.3.7 Corollary

If V is a finite dimensional $\underline{g}$-module, then $\mathrm{End}_F(V)$ is a g-module via

$$(x.f)(v) = x.f(v) - f(x.v)$$

2.3.8 Definition

Let g be a Lie algebra over K. We call the **derived series of** g the following sequence of ideals of g defined by : $g^{(0)} = g, g^{(1)} = [g, g], g^{(2)} = [g^{(1)}, g^{(1)}], \ldots, g^{(n)} = [g^{(n-1)}, g^{(n-1)}]$. We say that g is solvable if there exists an n such that $g^{(n)} = 0$.

2.3.9 Example

1. If g is abelian, then $[g, g] = g^{(1)} = 0$, and so g is solvable.
2. If g is simple, then $[g, g] = g$, so we deduce that $g^{(n)} = g$, for all n and so g is not solvable.

We have

2.3.10 Proposition

a. If g is solvable then any Lie subalgebra of g and any homomorphic image of g are also solvable.

b. If an ideal I of g is solvable and g/I is solvable then g is solvable (i.e. the class of solvable Lie algebras is closed under extensions).

c. The sum of solvable ideals of g is a solvable ideal of g.

Proof

a. First assertion follows from the fact that for a Lie subalgebra $h \subset g$ we have $h^{(i)} \subset g^{(i)}$ for $i \in \mathbb{N}$. Consider a surjective Lie algebra morphism $\phi : g \to h$. By induction on i we establish $\phi(g^{(i)}) = h^{(i)}$. This is clear for $i = 0$ since φ is surjective. Assume $\phi(g^{(i-1)}) = h^{(i-1)}$ then we have $\phi(g^{(i)}) = \phi([g^{(i-1)}, g^{(i-1)}]) = [h^{(-1)}, h^{(i-1)}] = h^{(i)}$.

b. Consider $m, n \in \mathbb{N}$ such that $I^{(m)} = 0$ and $(g/I)^{(n)} = 0$. Let $\pi : g \mapsto g/I$ be the canonical surjective morphism. From a. it follows that $\pi(g^{(n)}) = (g/I)^{(n)} = 0$, hence $g^{(n)} \subset I$ and then $(g^{(n)})^{(m)} = g^{(n+m} \subset I^{(m)} = 0$.

c. For ideals I and J of g we have that $(I + J)/I = I/I \cap J$ is solvable as a quotient of a solvable Lie algebra, since J is also solvable it follows from b. that $I + J$ is solvable. □

2.3.11 Corollary

The sum of all solvable ideals is a solvable ideal, it is the biggest solvable ideal of g and will be denoted by $\mathrm{Rad}(g)$, called the **radical of** g.

Proof As g is finite dimensional the sum of all solvable ideals is a finite sum and we may apply c. of Proposition 2.3.10. □

2.3.12 Definition

g is said to be **semisimple** if $\mathrm{Rad}(g) = 0$.

2.3.13 Example

1. If $g \neq 0$ is semisimple then g is not solvable. If g is solvable then $g = \mathrm{Rad}(g)$ and so g is not semisimple.

2. If g is simple then g is semisimple. Indeed, $[g, g] = g$, so g is not solvable thus $g \neq [g, g]$ and simplicity of g then yields $\mathrm{Rad}(g) = 0$.

3. For every $g, g/\mathrm{Rad}(g)$ is semisimple. Indeed, if $g/\mathrm{Rad}(g)$ is not semisimple then there exists a nonzero solvable ideal $I(\mathrm{Rad}(g)$ where I is an ideal containing $\mathrm{Rad}(g)$. From the exact sequence $0 \to \mathrm{Rad}(g) \to I \to I/\mathrm{Rad}(g) \to 0$ it follows that I is solvable (cf. Proposition 2.3.10(b)) contradicting the fact that $\mathrm{Rad}(g)$ is the biggest solvable ideal. □

The representation theory of solvable Lie algebras may be described rather easily (see Lie's theorem) but the theory of representations of semisimple Lie algebras is far more difficult. For an introduction to representation theory we refer to Humphreys [10], or Fulton, Harris [6]. It is clear that by 3. above the representation theory of any finite dimensional Lie algebra may be "reduced" to the solvable and semisimple case. Observe that a subalgbra of a semisimple one need not be semisimple for example $\mathrm{sl}_2(K)$ is simple hence semisimple but it has 1-dimensional subalgebras which are abelian hence solvable and not semisimple. An ideal of a semisimple algebra however will be semisimple.

The **central descending series** of a Lie algebra g is defined by : $g^0 = g, g^1 = [g, g], g^2 = [g^1, g], \ldots, g^n = [g^{n-1}, g]$.

2.3.14 Definition

We say that g is **nilpotent** if $g^i = 0$ for some $i \in \mathbb{N}$.

2.3.15 Example

1. If g is abelian then it is nilpotent.
2. Since $g^{(i)} \subset g^i$, if g is nilpotent then it is solvable.

Now if I is an ideal of g such that I and g/I are nilpotent then g need not be nilpotent (construct a counter example !) but we do have :

2.3.16 Proposition

a. If g is nilpotent then any Lie subalgebra and any homomorphic image are nilpotent.

b. By $Z(g)$ we mean the subalgebra $\{x \in g, [x, y] = 0 \text{ for all } y \in g\}$ and call it the **centre** of g. If $g/Z(g)$ is nilpotent then g is nilpotent.

c. If g is nilpotent and $g \neq 0$, then $Z(g) \neq 0$.

Proof

a. Exactly as in Proposition 2.3.10. a.

b. Choose n such that $(g(Z(g))^n = 0$ and let $\pi : g \to g/Z(g)$ be the canonical surjective morphism. From $\pi(g^n) = (g/Z(g))^n$ it follows that $g^n \subset Z(g)$ and thus $g^{n+1} \subset [g, Z(g)] = 0$.

c. If n is such that $g^n \neq 0$ and $g^{n+1} = 0$ then it follows that $0 \neq g^n \subset Z(g)$.

2.3.17 Exercise

Prove that g is solvable, resp. nilpotent, if and only if ad g is solvable, resp. nilpotent.

Consider a finite dimensional vector space V over K, and $u \in \mathrm{End}_K(V)$ with associated matrix A in a fixed K-basis for V. If λ is an eigenvalue of u, that is a root of the characteristic polynomial $\det(TI - A)$, then the system $(\lambda I - A)x = 0$ has a nonzero solution (an eigenvalue of u). If a is nilpotent then u is a root of a polynomial T^n, hence the minimal polynomial of u is T^k. It follows that the only eigenvalue of a nilpotent u is 0 and u has an eigenvector. The following result extends on these remarks.

2.3.18 Theorem

Let $V \neq 0$ be a finite dimensional vector space and g a Lie subalgebra of $\mathrm{gl}(V)$. If all elements of g are nilpotent then they have a common eigenvector $v \in V, v \neq 0$ such that $g.v = 0$ (that is $x.v = 0$ for all $x \in g$).

Proof The argument is an induction on $\dim_K g$. If $\dim_K g$ is 0 or 1 then the statement is clear by the preceding remarks. So assume $\dim_K g = n > 1$ and the assertion holds for Lie algebras of dimension less than n. Consider a proper subalgebra h of g. For $x \in h$ the K-linear ad $x : g \to g, y \mapsto [x, y]$ leaves h globally invariant and it induces a map $\overline{\mathrm{ad}x} : g|h \to g|h$, $y + h \mapsto [x, y] + h$. So we obtain a Lie algebra morphism $\overline{\mathrm{ad}} : h \to \mathrm{gl}(g|h), x \mapsto \overline{\mathrm{ad}x}$. The image of $\overline{\mathrm{ad}}$ is a Lie subalgebra of $\mathrm{gl}(g/h)$ of dimension strictly less than the one of h and thus less than $\dim_K g$. Since every $x \in h$ is nilpotent we have that ad x is nilpotent and $\overline{\mathrm{ad}x}$ too. By the induction hypothesis there exists $y + h, y \in g/h$ such that $\overline{\mathrm{ad}x}(y) = \overline{o} = h$, for all $x \in h$. Thus it follows that $[y, h] \subset h$ and also $y \in N_g(h) - h$ where $N_g(h)$ stands for the normalizer of h in g i.e. the set $\{y \in g, [y, H] \subset H\}$. Now choose h to be a maximal proper Lie subalgebra of g. By the above $N_g(h) = g$ and h is an ideal of g. If $\dim_K(g/h) > 1$ then letting g' be a 1-dimensional Lie subalgebra of g/h we would have $0 \subsetneqq g' \subsetneqq g/h$ and then $h \subsetneqq \pi^{-1}(g') \subsetneqq g$, where $\pi : g \to y/h$ is the canonical surjective morphism. The latter would contradict the maximality of h, hence h is of codimension 1 in g. The induction hypothesis applied to $h \subsetneqq \mathrm{gl}(V)$ learns that $W = \{v \in V, h.v = 0\} \neq 0$. The K-subspace W is left invariant for any $x \in g$; indeed if we look at $w \in W$ and $y \in h$ then $[y, x] \in h$, hence $yx.w = xy.w + [y, x].w = x.0 + 0 = 0$, thus $g.W \subset W$. Finally, a $z \in g/h$ has an eigenvector $v \in W$ (by the 1-dimensional case, as $\dim_K g/h = 1$). Then we obtain $g.v = 0$ and this concludes the proof. □

We are now ready for the following theorem.

2.3.19 Theorem (Engel)

The following conditions are equivalent.

1. g is nilpotent.

2. All elements are ad-nilpotent where $x \in g$ is said to be **ad-nilpotent** if ad$x$ is nilpotent i.e. $(\mathrm{ad}x)^n = 0$.

Proof The implication $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 1$ The assumption yields that $\mathrm{ad}g \subset \mathrm{gl}(g)$ satisfies the hypothesis of foregoing Theorem 2.3.15. Thus there exists an $x \in g$, $x \neq 0$ such that $[g, x] = 0$ and therefore $Z(g) \neq 0$. Therefore the dimension of $g/Z(g)$ is strictly less than $\dim_K g$. Since $Z(g)$ is an ideal of g it is left invariant by $\mathrm{ad}x$ for all $x \in g$, consequently every endomorphism $\mathrm{ad}x : g \to g$ induces $\mathrm{ad}x : g/Z(g) \to g/Z(g)$ which is also nilpotent. By induction on $\dim_K g$ it follows that $g/Z(g)$ is nilpotent and therefore g is nilpotent in view of Proposition 2.3.16.b. □

2.3.20 Exercise

Show that $\mathrm{sl}_2(K)$ is nilpotent if $\mathrm{ch}(K) = 2$.

2.3.21 Definition

Let V be a K-vector space of dimension n. A **flag** in V is a sequence of subspaces : $0 = V_0 \subset V_i \subset \ldots \subset V_n = V$, such that $\dim_K V_i = i, i = 0, 1, \ldots, n$.

Consider $x \in \mathrm{End}_K V$. Then there exists a flag (V_i) such that $x.V_i \subset V_{i-1}$, for all i, if and only if x is nilpotent. Indeed, the existence of such a flag means that there exists a K-basis $\{v_1, \ldots, v_n\}$ for V such that the matrix associated to x is strictly upper triangular, i.e.

$$\begin{aligned}
&x.v_1 = 0\\
&x.v_2 = a_{12}v_1\\
&x.v_3 = a_{13}v_1 + a_{23}v_2\\
&\quad\vdots\\
&x.v_n = a_{1n}v_1 + \ldots + a_{n-1}v_{n-1}
\end{aligned}$$

It clearly follows then that x is nilpotent. Conversely if x is nilpotent look at an eigenvector v_1 for $x, v_1 \neq 0, x.v_1 = 0$. It follows that x includes a morphism : $V/Kv_1 \to V/Kv_1$, which is also nilpotent. Thus there exists an eigenvector v_2, etc...; we repeat this process untill we have obtained a flag (V_i) such that $x.V_i \subset V_{i-1}$.

2.3.22 Corollary

With hypothesis as in Theorem 2.3.18, there exists a flag (V_i) in V such that $g.V_i \subset V_{i-1}$ for all i.

Proof Consider $v \neq 0$ in V such that $g.v = 0$. Put $V_1 = Kv$ and $W = V/V_1$. Then g acts on W and by induction on $\dim_K V$ there exists a flag on W with the required property. To this there corresponds a flag in V by looking at the inverse images of the flag in W and adding V_1 at the beginning. □

2.3.23 Corollary

Let h be a nonzero ideal of g a nilpotent Lie algebra, then $h \cap Z(g) \neq 0$.

Proof Since h is an ideal, g acts on h via the adjoint representation $g \to g/(h)$. The image of this morphism ad$g$ satisfies the hypothesis of Theorem 2.3.18 and therefore there exists an $x \in h, x \neq 0$ such that $[g, x] = 0$. Hence $x \in Z(g)$ and the proof is complete. □

2.4 Semisimple Lie Algebras

In this section we assume that K is algebraically closed and $\mathrm{ch}(K) = 0$.

2.4.1 Theorem (Lie)

Let V be a finite dimensional vector space, $V \neq 0$, and g a solvable Lie subalgebra of $\mathrm{gl}(V)$. Then there exists a common eigenvector for all elements of g.

Proof The proof is by induction on $\dim_K g$. When $\dim_K g$ is 0 or 1 everything is clear (note that the assumption that K is algebraically closed is necessary even in case of dimension 1, the characteristic polynomial of the generator of g has at least one root and so there exists an eigenvector). We assume $\dim_K g > 1$ and the induction hypothesis.

We follow the plan of the proof of Engel's theorem :

Step 1. Find in g an ideal h of codimension 1.

Step 2. Use the induction hypothesis to find a common eigenvector for all elements of h.

Step 3. g stabilizes a subspace W of V, consisting of eigenvectors for the elements of h.

Step 4. Find in W an eigenvector for z where $g = h + Kz$.

Step 1 is easier, but Step 3. more difficult when compared to the proof of Engel's Theorem.

Step 1. Since g is solvable and $\dim_K g > 1$, it follows that $g \neq [g, g]$ and $g/[g, g]$ is abelian, hence any subspace of codimension 1. is an ideal. Take one of these and take its inverse image for $\pi : g \mapsto g/[g, g]$, the canonical surjective morphism. This leads to an ideal h with $\dim_K(g/h) = 1$ and $[g, g] \subset h$.

Step 2. Since h is solvable the induction hypothesis implies the existence of $v \in V$ which is an eigenvector for all $x \in h$, i.e. $v \neq 0$ and for all $x \in h, x(v) = \lambda(x)v$ where $\lambda : h \to K$ is linear.

Step 3. Write W for $\{w \in V, x(w) = \lambda(x)w, \text{ for all } x \in h\}$. Then $W \neq 0$ and we show now that $g.W \subset W$. Consider $x \in g$ and $w \in W$, we want $x(w) \in W$, that is for all $y \in h$ we have $yx(w) = \lambda(y)x(w)$. But $yx(w) = xy(w) - [x, y](w) = x\lambda(y)w - \lambda([x, y])w$ since $y, [x, y] \in h$. It is sufficient to show that for all $x \in g$, $y \in h$, we have that $\lambda([x, y]) = 0$. Let $w \in W$ and $n > 0$ the smallest such that $w, x(w), \ldots, x^n(w)$ are linearly independent (if $n = 0$ there is nothing to prove). Denote by W_i the subspace of V generated by $w, x(w), \ldots, x^{i-1}(w)$, $W_i =< w, x(w), \ldots, x^{i-1}(w) >$. By definition we put $W_0 = 0$. We have $\dim W_i = i$ for $i = 0, \ldots, n$, $W_{n+j} = W_n$ for all j and obviously $x(W_n) \subset W_n$.

Consider the K-basis of W_n, $\{x, x(w), \ldots, x^{n-1}(w)\}$. We have $y(W_n) \subset W_n$ and the matrix of $y|W_n$ in this basis is upper triangular and all diagonal elements equal $\lambda(y)$; this follows from $y(W_i) \subset W_i$, for all i, and $yx^i(w) \equiv \lambda(y)x^i(w) \mathrm{mod} W_i$, for all $y \in h$. Now we procede by induction on i, if $i = 0$ then everything is clear because $y(0) = 0$ and $y(w) = \lambda(y)w$ since $y \in h$. So assume that $y(W_j) \subset W_j$ for $j \leq i$ and $yx^i(w) = \lambda(y)w^i(w) + w'$ with $w' \in W_i$, for all $y \in h$. We compute for $y \in h$: $yx^{i+1}(w) - yxx^i(w) = xyx^i(w) - [y,x]x^i w = \lambda(y)x^{i+1} + x(w') - \lambda([y,x])x^i w - w''$, where $w', w'' \in W$ and we use the induction hypothesis plus $[y,x] \in h$. Since $\lambda(w') \in W_{i+1}$ and $-\lambda([y,x])x^i(w) - w'' \in W_{i+1}$, it follows that the assertion holds for $i+1$. Hence, for $y \in h$ we have $\mathrm{Tr}(y|W_n) = n\lambda(y)$. In particular, taking y of the form $[x, y'] \in h$ for x as above and $y' \in h$ and using the fact that x and y both stabilize W_n, it follows that $[x, y']$ is a commutator of two endomorphisms and so its trace is zero, i.e. $n\lambda([x, y']) = 0$. From $\mathrm{ch}(K) = 0$ it thus follows that $\lambda([x, y]) = 0$ for all $y \in h$.

Step 4. We have $g = h + Kz$ for some $z \in g - h$. Since K is algebraically closed there exists an eigenvector of z in W, so this is an eigenvector for all elements of g (and λ extends to a map in g^*).

2.4.2 Remarks

1. The n in Step 3. is at most $\dim_K V$, so the hypothesis $\mathrm{ch}K = 0$ may be replaced by $\mathrm{ch}(K) > \dim_K V$.
2. In the proof we used $g \neq [g, g]$ and not really the solvability of g. This condition is strictly weaker than solvability, $\mathrm{gl}_n(K)$ is not solvable but $[\mathrm{gl}_n(K), \mathrm{gl}_n(K)] = \mathrm{sl}_n(K)$. However algebras with the property $[g, g] \neq g$ are not closed under ideals, see also the foregoing example, hence the result of the theorem does not extend to this class of algebras.
3. In step 3. we cannot let $yx^i(w) = \lambda(y)x^i(w) - [x^i, y](w)$ (and we cannot get rid of the induction) because x^i is not necessarily an element of g and thus $[x^i, y] \notin h$.

2.4.3 Exercise

An irreducible representation of a solvable Lie algebra is one dimensional.

2.4.4 Corollary (Lie)

Let V be a finite dimensional K-space and g a solvable Lie subalgebra of $\mathrm{gl}(V)$. There exists a flag on V whose subspaces are stabilized by all elements of g.

Proof By induction on $\dim_K V$. If $\dim_K V = 1$ then the statement is obvious. If $\dim_K(V) > 1$ we let v be an eigenvector for all elements of g and we put $V_1 = Kv$. Then V_1 is a subspace stabilized by all elements of g so g acts on V/V_1. The image of g in $\mathrm{gl}(V/V_1)$ is also solvable so by the induction hypothesis it stabilizes a flag in V/V_1. Take the preimage of this flag in V starting it with V_1 yields a flag in V. □

2.4.5 Corollary

If g is solvable then there exists a flag of ideals in g.

Proof Consider the adjoint representation ad : $g \to \mathrm{gl}(g)$. Since g is solvable ad$g \subset \mathrm{gl}(g)$ is also solvable and the foregoing corollary then states that it stabilizes a flag. Now a subspace of g stabilized by ad$g$ is an ideal.

□

2.4.6 Corollary

g is solvable if and only if $[g, g]$ is nilpotent.

Proof Suppose g is solvable. There exists a basis in which the matrices of ad$g$ are uppertriangular; so the matrices of $\mathrm{ad}_g[g, g] = [\mathrm{ad}g, \mathrm{ad}g]$ are strictly upper triangular. Consequently, for all $x \in [g, g], \mathrm{ad}_g x$ is nilpotent so $\mathrm{ad}_{[g,g]}x$ is nilpotent and then $[g, g]$ is nilpotent by Engel's theorem. Conversely $[g, g]$ is nilpotent hence solvable then by definition g is solvable.

□

Let V be a finite dimensional K-space, $u : V \to V$ an endomorphism. Then V becomes a finitely generated $K[T]$-module, where T is a variable, via the K-linear morphism $K[T] \to \mathrm{End}_K V, T \mapsto u$, i.e. $T.v = u(v)$ for $v \in V$. We may decompose V as a $K[T]$-module as a sum of cyclic submodules $K[T]x_i$. If u can be diagonalized, let μ_i be the elementary divisors of the x_i, then $\dim_K K[T]x_i = \deg\mu_i = 1$ and thus in this case V as a $K[T]$-module is semisimple. This is the basis for the following definition.

2.4.7 Definition

An endomorphism of V, a finite dimensional K-space, is said to be **semisimple** if and only if there exists a K-basis for V in which its matrix is diagonal i.e. the basis consists of eigenvectors of the endomorphism.

We recall from linear algebra the following facts :

2.4.8 Properties

1. Let V and u be as above, then the following statements are equivalent.

   a. u is semisimple.

   b. There is a v with $n = \dim_K V$ distinct eigenvalues commuting with u.

   c. The minimal polynomial of u has only simple roots.

2. Two semisimple endomorphisms which commute may be diagonalized simultaneously.

3. The sum of two semisimple endomorphisms which commute is also semisimple.

2.4.9 Proposition

Consider $x \in \mathrm{End}_K V$, then the following hold :

a. There exist uniquely determined $x_s, x_n \in \mathrm{End}_K(V)$ such that $x = x_n + x_s$ where x_s is semisimple and x_n is nilpotent and x_s and x_n commute.

b. There exist $p(T), q(T)$ in $K[T]$ without constant term such that $x_s = p(x), x_n = q(x)$. In particular x_s and x_n commute with every endomorphism that commutes with x.

c. If $A \subset B \subset V$ are subspaces such that $xB \subset A$ then the same holds for x_s and x_n.

Proof

a. We know there exists a basis for V in which x has a matrix with blocks of following type on the diagonal :

$$\begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & & & a \end{pmatrix}$$

Let x_s be the endomorphism corresponding to the diagonal of x and put $x_n = x - x_s$ we provice a direct proof for a. and b. Consider the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of x, let $P_k(T) = \prod_{i=1}^{k}(T - \lambda_i)^{m_i}$ be the characteristic polynomial of x. Put : $V_i = \mathrm{Ker}(x - \lambda_i.I)^{m_i}$. Then we have $V = \oplus_{i=1}^{k} V_i$ and each V_i is stabilized by x with $\dim_K V_i = m_i$. The characteristic polynomial of $x|V_i$ is $(T - \lambda_i)^{m_i}$. First we establish $V = \sum_{i=1}^{k} V_i$. Put $f_i = P_x(T)/(T - \lambda_i)^{m_i}$, then $(f_1, \ldots, f_k) = 1$ so there exist by Bezout's theorem polynomials g_i suh that $\sum f_i g_i = 1$. Thus we have $\sum f_i(x)g_i(x) = I_V$ and if $v \in V$ then $v = \sum_{i=1}^{k} v_i$ where $v_i = f_i(x)g_i(x)v \in V_i$. Next we show that $\sum V_i$ is a direct sum, so assume (up to reordering) that $v_1 \in V_1 \cap \sum_{i=2}^{k} V_i$, $v_1 = v_2 + \ldots + v_k$ with $v_i \in V_i$. Consider $R = \prod_{i=2}^{k}(T - \lambda_i)^{m_i}$ and $P = (T - \lambda_1)^{m_1}$. We have that $(R, P) = 1$ and $R(x)v_1 = 0$, $P(x)v_1 = 0$ hence $v_1 = 0$. Obviously x commutes with $(x - \lambda_i I)^{m_i}$ hence $x(V_i) \subset V_i$. Observe that the characteristic polynomial of $x|V_i$ is of the form $(T - \lambda_i)^{k_i}$. Since it does not depend on the basis we may choose in each V_i a basis in which x_n is strictly upper triangular, x_s diagonal still. The basis of V obtained this way leads to $\prod_{i=1}^{k}(T - \lambda_i)^{k_i} = \prod_{i=1}^{k}(T - \lambda_i)^{m_i}$ hence $k_i = m_i$ and the statement concerning the characteristic polynomial of $x|V_i$ is clear once we have the property of x_n.

Now choose $p(T)$ and $q(T)$ as follows, applying the Chinese remainder theorem : $p(T) \equiv \lambda_i \mathrm{mod}(T - \lambda_i)^{m_i}$ and $p \equiv 0 \ \mathrm{mod} T$ (if 0 is not an eigenvalue), $q(T) = T - p(T)$ and $x_s = p(x), x_n = q(x)$.

Then x_s is semisimple because $x_s(v) = \lambda_i v$ for all $v \in V_i$, for all i. On the other hand x_n is nilpotent because it is nilpotent on every V_i, i.e. for $v \in V_i$, $x_n^{m_i} v = (x - x_s)^{m_i} v = (x - \lambda_i I)^{m_i} v = 0$ by definition of V_i.

If $x = x_s + x_n = s + n$ where s is semisimple and n nilpotent and s and n commute, then s and n commute with x thus by the above they also commute with x_s and x_n. Then $x_s - s = n - x_n$ is both semisimple and nilpotent so it is zero.

c. Obvious from $x_s = p(x), x_n = q(x)$. □

We say that an element is **semisimple** when *ad* applied to that element is semisimple, similarly an element is **nilpotent** if it is *ad*-nilpotent. We call $x = x_s + x_n$ the **Jordan decomposition** of x and x_s is the **semisimple** part x_n the **nilpotent part** of x.

2.4.10 Lemma

Let $x \in \mathrm{End}_K V$ have Jordan decomposition $x = x_s + x_n$, then $\mathrm{ad}x = \mathrm{ad}x_s + \mathrm{ad}x_n$ is the Jordan decomposition of $\mathrm{ad}x$ in $\mathrm{End}_K(\mathrm{End}_K V)$.

Proof We know that $\mathrm{ad}x_n$ is nilpotent (hence ad-nilpotent) so it remains to show that $\mathrm{ad}x_s$ is semisimple and commutes with x_n. Select a K-basis $B = \{v_1, \dots, v_n\}$ such that the matrix of x in B is $M_B(x) = \mathrm{diag}(a_1, \dots, a_n)$. In $\mathrm{End}_K(V)$ consider the basis $\{e_{ij}\}$ and compute : $\mathrm{ad}x(e_{ij}) = M_B(x).e_{ij} - e_{ij}M_B(x) = (a_i - a_j)e_{ij}$. Therefore $\mathrm{ad}x$ is semisimle.
Finally we have :$[\mathrm{ad}x_s, \mathrm{ad}x_n] = \mathrm{ad}[x_s, x_n] = 0$, which yields the commutation property for $\mathrm{ad}x_s$ and $\mathrm{ad}x_n$. □

2.4.11 Lemma

Let A be a K-algebra (not necessarily associative) and $\delta \in \mathrm{Der}_K A \subset \mathrm{End}_K A$. Then $\delta_s, \delta_n \in \mathrm{Der}_K A$.

Proof Put $\delta = \delta_s + \delta_n$ it is enough to prove that $\delta_s \in \mathrm{Der}_K A$. For $a \in K$ put $A_a = \{x \in A,$ there is a k such that $(\delta - a.I)^k x = 0\}$. Observe that δ and δ_s have the same eigenvalues (decompose W into a direct sum of eigenspaces for δ_s and choose bases in which δ_n has strictly upper triangular matrices).

If a is not an eigenvalue for δ then $A_a = 0$, and if a is an eigenvalue then δ_s acts diagonally on A_a, $\delta_s(x) = ax$ for all $x \in A_a$. Then we have $A = \oplus_{a \in K} A_a$. Now $A_a A_b \subset A_{a+b}$ follows from :

$$(\delta - (a+b).I)^n(xy) = \sum_{i+j=n} C_n^i (\delta - aI)^i(x)(\delta - bI)^j(y)$$

which can be proved by induction (straightforward). If $x \in A_a, y \in A_b$, then $xy \in A_{a+b}$ and we have : $\delta_s(xy) = (a+b)xy = axy + bxy = \delta_s(x)y + x\delta_s(y)$. Now, for $x, y \in A$, say $x = \Sigma' x_a, y = \Sigma' y_b$ (Σ' denotes finite sums), we have :

$$\delta_s(xy) = \Sigma'\delta_s(x_a y_b) = \Sigma'\delta_s(x_a)y_b + x_a\delta_s(y_b) = \delta_s(x)y + x\delta_s(y)$$

and therefore δ_s is a derivation and so is $\delta - \delta_s = \delta_n$. □

We will now derive Cartan's solvability criterion from the Jordan-Chevalley decomposition. Such a criterion for solvability is equivalent to a criterion for nilpotency of $[g, g]$ and by Engel's theorem it is enough to give a criterion for $\mathrm{ad}x$ to be nilpotent for all $x \in [g, g]$. We will show that for $x \in [g, g]$, $(\mathrm{ad}x)_s = 0$.

2.4.12 Lemma

Let $A \subset B \subset \mathrm{gl}(V)$ be K subspaces and put $M = \{x \in \mathrm{gl}(V), [x, B] \subset A\}$. For $x \in M, \mathrm{Tr}(xy) = 0$ for all $y \in M$ entails that x is nilpotent.

Proof Let $x = s + n$ be the Jordan decomposition and B a K-basis such that

$$M_B(s) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{pmatrix}$$

Let E be the $\mathbb{Q}$-subspace of K generated by $a_1, \ldots, a_m \in K$. We want $E = 0$ or $E^* = 0$ where $E^* = \mathrm{Hom}_Q(E, \mathbb{Q})$. Pick $f \in E^*$ and let $y \in \mathrm{gl}(V)$ have matrix $M_B(y)$,

$$M_B(y) = \begin{pmatrix} f(a_1) & & 0 \\ & \ddots & \\ 0 & & f(a_n) \end{pmatrix}$$

Considering the basis $\{e_{ij}\}$ in $\mathrm{End}_K V$ we calculate : $\mathrm{ad}s(e_{ij}) = (a_i - a_j)e_{ij}$ and $\mathrm{ad}y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$. Now use Lagrange interpolation.

2.4.13 Exercise (Lagrange interpolation)

If $a_1, \ldots, a_{k+1} \in K$ are distinct and $c_1, \ldots, c_{k+1} \in K$, then there exists a polynomial $P \in K[T]$ of degree k such that $P(a_i) = c_i$, $i = 1, \ldots, k{+}1$. So let $R(T) \in K[T]$ be the polynomial without constant term such that $R(a_i, ..., a_j) = f(a_i) = -f(a_j)$ (we eleminate repetitions, correctness is ensured by the K-linearity of f, i.e. if $a_i - a_j = a_k - a_l$ then $f(a_i) - f(a_j) = f(a_k) - f(a_l)$). Since $R(\mathrm{ad}s)$ and $\mathrm{ad}y$ coincide on the basis $\{e_{ij}\}$ we have $R(\mathrm{ad}s) = \mathrm{ad}y$. Now $\mathrm{ad}s$ is a polynomial without constant term in $\mathrm{ad}x$, hence $\mathrm{ad}y$ is a polynomial without constant term in $\mathrm{ad}x$. Since $\mathrm{ad}x(B) \subset A$ it follows that $\mathrm{ad}y(B) \subset A$ too, or $y \in M$. By the hypothesis we have $\mathrm{Tr}(xy) = 0$ and by computing the trace in the basis B we obtain $\Sigma' a_i f(a_i) = 0$. Applying f to the latter yields $\sum f(a_i)^2 = 0$ and the $f(a_i) \in \mathbb{Q}$, thus $f(a_i) = 0$ for all i, hence $f = 0$. Thus $E^* = 0$ or $E = 0$. □

2.4.14 Exercise

For $x, yz \in \mathrm{gl}(V)$ we have : $\mathrm{Tr}([x, y]z) = \mathrm{Tr}(x[y, z])$.

2.4.15 Theorem (Cartan's criterion)

Consider $g \subset \mathrm{gl}(V)$ a Lie subalgebra such that $\mathrm{Tr}(xy) = 0$ for all $x \in [g, g]$, $y \in g$, then g is solvable.

Proof We show all elements of $[g, g]$ are nilpotent (hence also ad-nilpotent). Put $A = [g, g], B = g$ and apply Lemma 2.4.12. Thus $M = \{x \in \mathrm{gl}(V), [x, g] \subset [g, g]\}$.

Clearly $g \subset M$. We know $\mathrm{Tr}(xy) = 0$ for all $x \in [g, g]$ and $y \in g$. Consider $\Sigma[x, y] \in [g, g]$ and $z \in M$. We have : $\mathrm{Tr}(\Sigma[x, y]z) = \Sigma\ \mathrm{Tr}([x, y]z) = \Sigma\mathrm{Tr}(x[y, z])$, because of Exercise 2.4.14. Since $z \in M, y \in g$, we have $[y, z] \in [g, g]$, hence the foreging also equals $\Sigma\ \mathrm{Tr}([y, z]x) = 0$. Again by Lemma 2.4.12 it follows that $\Sigma[x, y]$ is nilpotent and as it was arbitrary chosen in $[g, g]$ the latter is nilpotent. □

2.4.16 Exercise

Prove the converse to Theorem 2.4.15.

2.4.17 Corollary

Let g be a Lie algebra such that $\mathrm{Tr}(\mathrm{ad}x\mathrm{ad}y) = 0$ for all $x \in [g, g], y \in g$, then g is solvable.

Proof Consider the adjoint representation $g \to \mathrm{gl}(g)$. Its image ad g is solvable. Indeed, if $x \in [\mathrm{ad}g, \mathrm{ad}g] = \mathrm{ad}[g, g]$ and $y \in \mathrm{ad}g$, then $x = \mathrm{ad}x'$ for some $x' \in [g, g]$, $y = \mathrm{ad}y'$ for some $y' \in g$ and we then calculate : $\mathrm{Tr}(xy) = \mathrm{Tr}(\mathrm{ad}x', \mathrm{ad}y') = 0$, so g is solvable by Theorem 2.4.15. Now $\mathrm{Ker}(\mathrm{ad}) = Z(g)$ is abelian hence solvable, thus g is solvable.

We now use traces to introduce a symmetric bilinear form $\mathcal{K}$ on g, called the Killing form.

2.4.18 Definition

The **Killing form** of g over K is given by : $\mathcal{K} : g \times g \to K$, $\mathcal{K}(x, y) = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y)$. It is by definitioin symmetric and K-bilinear.

2.4.19 Exercise

$\mathcal{K}$ is associative, i.e. $K([x, y], z) = \mathcal{K}(x, [y, z])$.

2.4.20 Exercise

Let $\varphi \in \mathrm{End}_K V, W$ a subspace of V such that $\varphi(V) \subset W$. Then $\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi|W)$.

2.4.21 Lemma

Let I be an ideal of g, $\mathcal{K}$ the Killing form of g and K_I the Killing form of I. Then $\mathcal{K}_I = \mathcal{K}|I \times I$.

Proof For $x, y \in I$ ad$x$ad$y : g \to g$ maps g to I for $(x, y) \in I \times I$ we have :

$$\begin{aligned}
\mathcal{K}(x,y) &= \mathrm{Tr}(\mathrm{ad}x\mathrm{ad}y) \\
&= \mathrm{Tr}(\mathrm{ad}x\mathrm{ad}y|I) \text{ (by Exercise 2.4.20)} \\
&= \mathrm{Tr}(\mathrm{ad}_I(x)\mathrm{ad}_I y) \text{ (since } I \text{ is an ideal)} \\
&= \mathcal{K}_I(x,y) \qquad \square
\end{aligned}$$

We define the radical of $\mathcal{K}$ by $S = \{x \in g, \mathcal{K}(x,y) = 0 \text{ all } y \in g\}$. Since $\mathcal{K}$ is K-bilinear S is a K-subspace of g and even an ideal because $\mathcal{K}$ is associative (for $x \in S, y \in g$ we have $\mathcal{K}([x,y],z) = \mathcal{K}(x,[y,z]) = 0$ for all $z \in g$). We say that $\mathcal{K}$ **is non-degenerate** if and only if $S = 0$, if and only if the determinant of the matrix of $\mathcal{K}$ in some basis of g is nonzero.

2.4.22 Example

Let $g = \mathrm{sl}_2(K)$ and $\{h.x.y\}$ a K-basis with $[x,y] = h, [h,x] = 2x.[h,y] = -2y$. Then the matrix of ad$h$ in this basis is :

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The matrix of ad$x$ in this basis is :

$$\begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix of ad$y$ in this basis is :

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Consequently, the matrix of $\mathcal{K}$ in this basis is :

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

Which has determinant $-128 = -2^7$. Thus $\mathcal{K}$ is non-degenerate if and only if $\mathrm{ch}(K) \neq 2$. On the other hand if $\mathrm{ch}(K) = 2$ then $\mathrm{sl}_2(K)$ is nilpotent (hence not semisimple). Thus $\mathrm{sl}_2(K)$ is semisimple if and only if $\mathcal{K}$ is nondegenerate.

2.4.23 Exercise

Show that g is semisimple if and only if g has no abelian ideals different from zero.

2.4.24 Theorem

The Lie algebra g is semisimple if and only if $\mathcal{K}$ is nondegenrate.

Proof ($\Longrightarrow$) Suppose $\mathrm{Rad}(g) = 0$ then we want $S = 0$ and that will follow if we establish that S is solvable. Let $x \in S$, $y \in [S,S]$, then $\mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y) = 0$ by definition of S. Since S is an ideal, lemma 2.4.21. entails : $0 = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y) = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y|S) = \mathrm{Tr}(\mathrm{ad}_S x\ \mathrm{ad}_S y)$.

By Cartan's criterion S is solvable.

($\Longleftarrow$) Assume $S = 0$, that is $\mathcal{K}$ is nondegenerate. From Exercise 2.4.23. it follows that we can finish the proof by showing that any abelian ideal I of g is contained in S. Consider such I, $x \in I$ and $y \in g$. Then $x \in S$ will follow if $\mathcal{K}(x,y) = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y) = 0$. For $z \in g$ we have : $(\mathrm{ad}x\ \mathrm{ad}y)^2(z) = [x,[y,[x,[y,z]]]] = 0$ since I is abelian and x, $[x,[y,z]],[y,[x,[y,z]]]$ are in I. Thus $\mathrm{ad}x\ \mathrm{ad}y$ is nilpotent and therefore its trace is zero. □

From the proof of the theorem it follows that $S \subset \mathrm{Rad}(g)$ but the converse inclusion need not hold.

A Lie algebra g is the direct sum of ideals $I_1, \ldots, I_k$ if and only if $g = I_1 \oplus \ldots \oplus I_k$ as K-vector spaces. It follows that $[I_i, I_j] \subset I_i \cap I_j = 0$, thus g may be obtained from the Lie algebras $I_1, \ldots, L_k$ by defining the bracket component-wise in the external direct sum $I_1 \oplus \ldots \oplus I_k$.

2.4.25 Theorem

Let g be a semisimple Lie algebra, then the following assertions hold :

1. There exist ideals $I_1, \ldots, I_k$ which are simple (as Lie algebras) such that $g = I_1 \oplus \ldots \oplus I_k$.
2. An ideal of a semisimple Lie algebra is again semisimple and so is every homomophic image of g. Any ideal of an ideal of g is an ideal of g and every ideal of g is a sum of simple ideals of g.
3. Any simple ideal of g coincides with one of the I_i.
4. We have $g = [g,g]$.
5. The Killing form $\mathcal{K}_i$ of I_i is the restriction of the Killing form $\mathcal{K}$ to $I_i \times I_i$ for all i.

Proof

1. & 2. If we establish that an ideal of g is a direct summand of g, then the proof will follow easily by induction. Let I be an ideal of g, $I^\perp = \{x \in g, \mathcal{K}(x,y) = 0$, for all $y \in I\}$. Take $x \in I^\perp, y \in g$; we want to show that $I^\perp$ is an ideal, i.e. $[x,y] \in I^\perp$, or, for all $z \in I$ we have $\mathcal{K}[[x,y],z] = 0$. But $\mathcal{K}([x,y],z] = K(x,[y,z]]) = 0$ since $x \in I^\perp, [y,z] \in I$. To obtain that $I \cap I^\perp = 0$ it suffices to show that $I \cap I^\perp$ is solvable by using Corollary 2.4.17. So we want that $\mathcal{K}(x,y) = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y) = 0$ for $x \in [I \cap I^\perp, I \cap I^\perp]$ and $y \in I^\perp$.

The latter is clear from the definition of $I^\perp$, thus $I \cap I^\perp$ is solvable and hence $I \cap I^\perp = 0$. Thus we obtain :

$$\begin{aligned} \dim_K(I + I^\perp) &= \dim_K I + \dim_K I^\perp - \dim_K(I \cap I^\perp) \\ &= \dim_K I + \dim_K I^\perp \end{aligned}$$

Let $\{e_1, \ldots, e_s\}$ be a K-basis for I and complete it to $\{e_1, \ldots, e_s,$

$e_{s+1}, \ldots, e_n\}$ a K-basis for g. Now $y \in I^\perp$ if and only if $\mathcal{K}(e_1, y) = \ldots = \mathcal{K}(e_s, y) = 0$, thus $I^\perp$ may be viewed as the set of solutions of a linear homogeneous system. The rank of the system is at most s, so the space of solutions has dimension n-rank $\geq n - s$. It follows then that : $\dim_K(I) + \dim_K(I^\perp) \geq s + (n - s) = n = \dim_K g$, hence $g = I \oplus I^\perp$. Now let I_1 be a minimal ideal of g. As before $g = I_1 \oplus I_1^\perp$ and $L_1^\perp$ is semisimple, because an ideal of I_1 is an ideal of g. So induction works (pass from g to $I_1^\perp$) and 1 as well as 2 have been proved.

3. Let $I \neq 0$ be a simple ideal of g. Then since $Z(g) = 0$, we have $[(I, g)] \neq 0$, but $[I, g] \subset I$, thus $[I, g] = I$ and : $I = [I, g] = [I, I_1] \oplus \ldots \oplus [I, I_k]$. Observe that the latter sum is indeed direct because $[I, I_i] \cap [I, I_j] \subset I_i \cap I_j = 0$. So for some i we have $[I, I_i] = I_i$ and the other terms are zero (as I is simple).

4. We have : $[g, g] = \oplus_i [g, I_i] = \oplus_i I_i = g$ (no $[g, I_i]$ is zero because that would contradict $Z(g) = 0$).

5. Obvious in view of Lemma 2.4.21. $\square$

2.4.26 Exercise

Give an example of a Lie subalgebra of a semisimple Lie algebra which is not semisimple. Give an example of an ideal of an ideal of a Lie algebra which is not an deal.

The set of inner derivations of a Lie algebra g, that is ad$g$, is an ideal of $\mathrm{Der}_K g$.

2.4.27 Theorem

If g is semisimple then every derivation of g is inner, in other words ad$g = \mathrm{Der}_K g$.

Proof Since g is semisimple, Ker(ad) $= Z(g) = 0$, hence ad$g \simeq g$ is semisimple too. Let $\mathcal{K}$ be the Killing form of $\mathrm{Der}_K g$ and take ad$g^\perp$, the orthogonal complement of ad$g$ with respect to $\mathcal{K}$. The Killing from of ad$g$ is $\mathcal{K}$ restricted to ad$g \times$ ad$g$ and it is nondegenerate in view of Theorem 2.4.24. We are in the following situation : V is a finite dimensional K-vector space, W a subspacion of V, $\mathcal{K}$ a symmetric bilinear form on V with restriction to $W \times W$ nondegenerate. We claim that $V = W \oplus W^\perp$ (internal direct sum). Indeed if $y \in W^\perp \cap W$ then $\mathcal{K}(x, y) = 0$ for all $x \in W$ and then $y = 0$ by the non-degenerency of $\mathcal{K}|W \times W$. Now continue as in the proof of 1. in Theorem 2.4.25. Going back to the particular situation we obtain $\mathrm{Der}_K g = \mathrm{ad}g \oplus \mathrm{ad}g^\perp$, and thus $[\mathrm{ad}g^\perp, \mathrm{ad}g] = 0$. Take $\delta \in \mathrm{ad}g^\perp$, then for all $x \in g$ we have that : $\mathrm{ad}(\delta(x)) = [\delta, \mathrm{ad}x] = 0$. Since ad is injective it follows that $\delta(x) = 0$ for all $x \in g$, or $\delta = 0$. Consequently if follows that $\mathrm{ad}g^\perp = 0$ and then $\mathrm{ad}g = \mathrm{Der}_K g$. $\square$

A representation $q : g \to \mathrm{gl}(V)$ is said to be **faithful** if Ker$q = 0$.

If g is semisimple and $q : g \to \mathrm{gl}(V)$ a faithful representation then we can define :

$$\beta : g \times g \to K, (x, y) \mapsto \mathrm{Tr}(q(x)q(y))$$

Then β is a bilinear symmetric form and β is associative since q is a Lie algebra map, therefore its radical :

$$S = \{x \in g, \beta(x, y) = 0 \text{ for all } y \in g\}$$

is an ideal of g. We have $\mathrm{Tr}(q(x)q(y)) = 0$ for all $q(x) \in q(S)$ and $q(y) \in [q(S), q(S)] \subset q[S]$. By Cartan's criterion $q(S)$ is solvable. But then $S \cong q(S)$ is solvable and since g is semisimple it then must follow that $S = 0$, i.e. β is nondegenerate. The Killing form is a particular case of the foregoing (take $q = \mathrm{ad}$).

So we fix a semisimple Lie algebra, β a bilinear symmetric associative and nondegenerate form on g. Let $\{e_1, \ldots, e_n\}$ be a K-basis in g.

2.4.28 Exercise

There exists a unique K-basis $\{f_1, \ldots, f_n\}$ in g such that $\beta(e_i, f_j) = \delta_{ij}$. The basis $\{f_1, \ldots, f_n\}$ is called the **dual basis** for $\{e_1, \ldots, e_n\}$.

Fix notation as above. For $x \in g$ write :

$$[x, e_i] = \sum_{j=1}^{n} a_{ij} e_j, \qquad [x, f_i] = \sum_{j-1}^{n} b_{ij} f_j)$$

The elements a_{ij} and b_{ij} are connected as follows :

$$\begin{aligned} a_{ik} &= \sum_{j=1}^{n} a_{ij}\delta_{jk} = \sum_{j=1}^{n} a_{ij}\beta(e_j, f_k) = \beta(\sum_{j=1}^{n} a_{ij}e_j,\ f_k) \\ &= \beta([x, e_i]f_k) = -\beta([e_i, x], f_k) = -\beta(e_i, [x, f_k]) \\ &= -\beta(e_i, \sum_{j=1}^{n} b_{kj} f_j) = -\sum_{j=1}^{n} b_{kj}\delta_{ij} = -b_{ki} \end{aligned}$$

Consider a representation $q : g \to \mathrm{gl}(V)$. Then we have $c_q(\beta) = \sum_{i=1}^{n} q(e_i)q(f_i)$ is in $\mathrm{End}_K V$ and it is an endomorphism of $q : g \to \mathrm{gl}(V)$. We want to show that $c_q(\beta)$ commutes with $q(x)$ for all $x \in \}$. Take $x \in g$ and compute :

$$\begin{aligned} [q(x), c_q(\beta)] &= \sum_{i=1}^{n} [q(x), q(e_i)q(f_i)] \\ &= \sum_{i=1}^{n} ([q(x), q(e_i)]q(f_i) + q(e_i)[q(x), q(f_i)]) \\ &= \sum_{i=1}^{n} (q([x, e_i])q(f_i) + q(e_i)q([x, f_i])) \\ &= \sum_{i=1}^{n} (q(\sum_{j=1}^{n} a_{ij}e_j)q(f_i) + q(e_i)q(\sum_{j=1}^{n} b_{ij}f_j)) \\ &= \sum_{i,j=1}^{n} (a_{ij}q(e_j)q(f_i) + b_{ij}q(e_i)q(f_j)) \\ &= \sum_{i,j=1}^{n} (-b_{ji}q(e_j)q(f_i) + b_{ij}q(e_i)q(f_j)) = 0 \end{aligned}$$

2.4.29 Definition

Let $q : g \to \mathrm{gl}(V)$ be a faithful representation of the semisimple Lie algebra g. Write $\beta(x, y) = \mathrm{Tr}(q(x)q(y))$ for the trace form of q. As above, β is nondegenerate. Fix a K-basis $\{e_1, \ldots, e_n\}$ of g and write c_q for $c_q(\beta)$; we call c_q the **Casimir element** for q. One easily calculates : $\mathrm{Tr}(c_q) = \mathrm{Tr}(\sum_{i=1}^n q(e_i)q(f_i)) = \sum_{i=1}^n \mathrm{Tr}(q(e_i)q(f_i)) = \sum_{i=1}^n \beta(e_i, f_i) = \sum_{i=1}^n \delta_{ii} = \dim_K g$.

2.4.30 Remark

If $q : g \to \mathrm{gl}(V)$ is irreducible then by Schur's lemma we obtain that c_q is a scalar. Since $\mathrm{Tr}(c_q) = \dim_K V = c_q = \dim_K g$, we obtain in this case that $c_q = \dim_K g / \dim_K V$. In this case the Casimir element does not depend on the chosen bases.

2.4.31 Exercise

Put $g = \mathrm{sl}_2(K), V = K^2$ and $q : g \to \mathrm{gl}V$ the identity map, compute c_q.

2.4.32 Exercise

Show that a semisimple Lie algebra g acts trivially on any 1-dimensional g-module.

The following result (Weyl's Theorem) is an analogue of Maschke's theorem for representations of finite groups.

2.4.33 Theorem (Weyl)

Let g be semisimple and $q : g \to \mathrm{gl}(V)$ a representation of g with V finite dimensional over K. Then $q : g \to \mathrm{gl}(V)$ is completely reducible.

Proof V is a finite dimensional g-module and we aim to show that any g-submodule W of V is a direct summand. If there is a g-module morphism $f : V \to W$ such that $f/W = I_W$ then it would follow that $\dim_K V = \dim_K(\mathrm{Ker} f) + \dim_K \mathrm{In}(f) = \dim_K(\mathrm{Ker} f) + \dim_K W, \mathrm{Ker} f \cap W = 0$ and thus $V = W \oplus \mathrm{Ker} f$. We now construct an f as desired. Look at $\mathrm{Hom}_K(V, W)$, which is a g-module via :

$$(x.f)(v) = x.f(v) - f(x.v)$$

(Proposition 2.3.6.). Denote

$$\mathcal{V} = \{f \in \mathrm{Hom}_K(V, W), f|W = aI_W \text{ for some } a \in K\}$$

If $f \in \mathcal{V}$ and $x \in g$, then for each $w \in W$ we have : $(x.f)(w) = x.f(w) - f(x.w) = x.aw - ax.w = 0$, hence $x.f|W = 0$ and thus $x.f \in \mathcal{V}$, or $\mathcal{V}$ is a g-module. Put $\mathcal{W} = \{f \in \mathcal{V}, f|W = 0\}$. This is a g submodule of $\mathcal{V}$ of codimension 1 as one observes in the commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{W} \arrow[r] \arrow[d, "I_{\mathcal{W}}"] & \mathcal{V} \arrow[r] \arrow[d, "I_{\mathcal{V}}"] & V/\mathcal{W} \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathcal{W} \arrow[r] & \mathcal{V} \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

(using the five lemma for exact sequences).

Assume that there is a g-submodule $\mathcal{X}$ of dimension 1 such that $\mathcal{V} = \mathcal{W} \oplus \mathcal{X}$ and let $\mathcal{X}$ be generated by f (we may assume $f|\mathcal{W} = I_{\mathcal{W}}$). Then for $x \in g$ we have $x.f = 0$. This means that for any $v \in V$ we have : $0 = (x.f)(v) = x.f(v) - f(x.v)$, thus $f(x.v) = x.f(v)$ and f is the desired g-module morphism.

Due to the foregoing the proof will be finished if we can show that any g-submodule W of V of codimension one has a complement. We will do this by induction on $\dim_K W$. Assume the assertion is true for all g-submodules of codimension one having dimension strictly smaller than $\dim_K W$. We may assume that W is irreducible indeed if W is not irreducible let $0 \neq W' \subset W$ be a proper g-submodule. Then : $0 \to W/W' \to V/W \to 0$ with $\dim_K W/W' < \dim_K W$, $\dim_K[(V/W')|(W/W')] = \dim_K V/W = 1$, hence we may apply the induction hypothesis to conclude that W/W' has a complement :

$$(*) \qquad V/W' = W/W' \oplus U/W'$$

On the other hand we have :

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W/W' \longrightarrow 0$$

with $\dim_K W' < \dim_K W$ and $\dim_K(W/W') = 1$. Again by the induction hypothesis W' has a complement, say

$$(**) \qquad W = W' \oplus X$$

We now show that $V = W \oplus X$. We have $\dim_K V = \dim_K W + 1 = \dim_K W + \dim_K X$, if $w \in W \cap X$ then $w \in W \cap U$ and so $w \in W'$ (use (*) above). Then $w = 0$ follows from (**). Now we have reduced to the situation where W is irreducible and of codimension one in V. Since $\mathrm{Ker}(q) = \mathrm{Ann}_g V$ and V has the same submodules whether regarded as a g-module or a $g/\mathrm{Ann}_g V$-module we may reduce further to the situation where $\ker q = 0$, i.e. q is faithful. Let $c : V \to V$ be the Casimir element of V. Now q induces a representation of V/W which is trivial by Exercise 2.4.32. This $g.V \subset W$ and also $c(V) \subset W$ (by definition of c). Thus $c(W) \subset W$ and $c|W$ is a scalar by Schur's lemma. Now the trace of $c|W$ equals the trace of c (see Exercise 2.4.20) and this is nonzero because it equals $\dim_K g$ (see Definition 2.4.29). The map $c : V \to W$ is a g-module morphism and $s|W = aI_W$ for some nonzero $a \in K$. We show that $\mathrm{Ker}(c) \cap W = 0$. Indeed if $w \in \mathrm{Ker}(c) \cap W$ then $c(w) = 0$ and since $w \in W$, $c(w) = aw$ thus $w = 0$.

It now suffices to take $f = a^{-1}c$, and the proof is finished. □

2.5 Right or wrong ?

1. A ring without nontrivial dervations is commutative.
2. If the composition of derivations is again a derivation then the composition is zero.
3. All $\mathbb{R}$-derivations of $\mathbb{C}$ are zero.

4. The $\mathbb{C}$-derivatios of $M_2(\mathbb{C})$ are a 3-dimensional $\mathbb{C}$-space.
5. If $[a_1, -]$ and $[a_2, -]$ are the same derivation of the K-algebra A, then $a_1 - a_2 \in K$.
6. A derivation of an algebra cannot be injective.
7. The K-derivations of a finite dimensional K-algebra form a finite dimensional K-vector space.
8. The inner derivations of an infinite dimensional noncommutative algebra always form an infinite dimensional K-vectorspace.
9. If $\mathrm{ch}(K) = 2$ then for every derivation of a K-algebra A and for every $a \in A$ we have $\delta(a^2) = 0$.
10. The derivations of a finite field over an arbitrary subfield are trivial.
11. The number of derivations of a finite algebra is a prime power.
12. If two algebras over the same field have isomorphic derivation Lie algebras, are these algebras then isomorphic ?
13. The commutator of an inner derivation and a non-inner derivation is again an inner derivation.
14. The kernel of a derivation is a subalgebra.
15. The derivations over a commutative algebra form a module over that algebra.
16. The derivations of a quotient of an algebra from a subset of the set of derivations of the algebra.
17. The derivations of the direct sum of two algebras can be written as sums of derivations of each algebra.
18. A derivation of a subalgebra can always be extended to a derivation of the original algebra.
19. If all derivations of an algebra are inner then this algebra has no zero divisors.
20. All derivations of $\mathbb{C}[x]$ are restrictions of inner derivations of the Weyl algebra $\mathbb{C} < x, y >$.
21. If $\mathrm{Der}_K A$ is an Abelian Lie algebra then A is commutative.
22. $\mathrm{Inn}_K A$ and A with the commutator bracket are isomorphic as Lie algebras.
23. Any two-dimensional Lie algebra is simple.
24. $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[X]/I)$ is finite dimensional for each proper ideal I of $\mathbb{C}[X]$.
25. If the K-algebra A is a field then every K-derivation of A is necessarily trivial.
26. Let δ_x be an inner derivation of the K-algebra A. If the map $[\delta_x, -] : \mathrm{Der}_K A \to \mathrm{Der}_K A, \delta \mapsto [\delta_x, \delta]$, is surjective, then $\mathrm{Der}_K A = \mathrm{Inn}_K A$.

27. Composition is never inner on $\mathrm{Der}_K A$.

28. A Lie algebra generated by one element is one dimensional.

29. Let g be a finite dimensional Lie algebra over $\mathbb{C}$. Is it true that for every $x \in g$, there is a y such that $[y, [x, y]] = 0$.

30. Let $\mathbb{H}$ be the quaternions over $\mathbb{R}$. Establish that $\mathrm{Der}_{\mathbb{R}}\mathbb{H} \cong \mathrm{so}_3(\mathbb{R})$.

31. A Lie algebra generated by two elements has dimension at most 3.

32. For $k + l = n$, $\mathrm{sl}_n(\mathbb{C}) = \mathrm{sl}_k(\mathbb{C}) \oplus \mathrm{sl}_l(\mathbb{C})$

33. The Lie algebra with structural constants λc_{ij}^k is isomorphic with a Lie algebra with structural constants c_{ij}^k.

34. If G and H are matrix groups with isomorphic Lie algebras then G and H are also isomorphic.

35. The upper triangular matrices form a simple Lie algebra.

36. Every complex finite dimensional simple Lie algebra contains $\mathrm{sl}_2(\mathbb{C})$ as a Lie subalgebra.

37. Every complex finite dimensional simple Lie algebra can be mapped surjectively to $\mathrm{sl}_2(\mathbb{C})$.

38. The enveloping algebra of a Lie algebra is never a free algebra.

39. The derivations of $\mathbb{C}^{\oplus n}$ form a simple Lie algebra.

40. The path algebra of a quiver of the form $0 \text{---} 0 \text{---} \cdots \text{---} 0 \text{---} 0$ is the enveloping algebra of a Lie algebra.

41. If $g \subset M_n(\mathbb{C})$ is a simple Lie algebra then all matrices in g have trace zero.

42. If every Lie subalgebra of g is a Lie ideal then g is Abelian.

43. If every Lie ideal of a non simple Lie algebra g is one dimensional, is then g Abelian ?

44. Each Lie algebra is a direct sum of simple Lie algebras.

45. If two matrices generate a two dimensional Lie algebra then one of the matrices is diagonal.

46. Every Lie subalgebra of $M_n(\mathbb{C})$ has dimension at most n.

47. The universal enveloping algebra of a finite dimension Lie algebra can be mapped to matrix algebra.

48. The universal enveloping algebra of a finite dimensional Lie algebra is never isomorphic to a matrix algebra.

Chapter 3

Filtered and Graded Rings

3.1 Group Graded Rings

We consider an arbitrary group G and write the multiplication for its operation, indicating it may be a non Abelian group. A ring R is said to be **G-graded** if there is a family of additive subgroups $\{R_\sigma, \sigma \in G\}$ of R such that $R = \oplus_{\sigma \in G} R_\sigma$ and $R_\sigma R_\tau \subset R_{\sigma\tau}$ for $\sigma, \tau \in G$. The elements of $\cup_{\sigma \in G} R_\sigma$ are called the **homogeneous elements** of R. A nonzero $x_\sigma \in R_\sigma$ is said to be **homogeneous of degree** σ. By definition every nonzero element r of R has a unique expression as a sum of homogeneous elements $r_\sigma \in R_\sigma, \sigma \in G$, say $r = \sum'_{\sigma \in G} r_\sigma$, where $\sum'$ denotes a finite sum.

3.1.1 Proposition

If R is a G-graded, e the neutral element of G, then R_e is a subring of R and $1 \in R_e$.

Proof From $R_e R_e \subset R_e$ it follows inmediately that R_e will be a subring of R if $1 \in R_e$. Let $1 = \sum_{\sigma \in G} r_\sigma$ be the homogeneous decomposition of $1 \in R$; pick $\tau \in G$ and $\lambda_\tau \in R_\tau$. The we calculate : $\lambda_\tau = 1.\lambda_\tau = \sum_{\sigma \in G} r_\sigma \lambda_\tau$, with $r_\sigma \lambda_\tau \in R_{\sigma\tau}$. Therefore, if $\sigma \neq e, r_\sigma \lambda_\tau = 0$ and $\lambda_\tau = r_e \lambda_\tau$, for all $\lambda_\tau \in R_\tau$ and for every $\tau \in G$. Hence $r_\sigma R = 0$ for $\sigma \neq e$ and $r_e \lambda = \lambda$ for every $\lambda \in R$, yields $1 = r_e \in R_e$. □

3.1.2 Examples

1. Let A be any ring, G a group and $(u_\sigma, \sigma \in G\}$ a set of symbols. The group ring $R = AG = \oplus R.a_\sigma$ with addition stemming from $Au_\sigma \simeq A$ and multiplication defined by : $(a_\sigma u_\sigma)(a_\tau u_\tau) = a_\sigma a_\tau u_{\sigma\tau}$ for all $\sigma, \tau \in G$, is G graded by taking $R_\sigma = Au_\sigma$.

2. The polynomial ring $K[X_1, \ldots, X_n] = R$ is $\mathbb{Z}$-graded by putting all X_i in degree 1, hence $R_0 = K$, $R_1 = KX_1 + \ldots + KX_n$, $R_m = R_1^m$ for all $m \in \mathbb{N}$. Since no negative degrees appear, we say that $K[X_1, \ldots, X_n]$ is **positively graded**.

3. Any ring R may be seen as a G-graded ring by taking$R_\sigma = 0$ for all $\sigma \neq e$ and $R_e = R$. In this case we say that R is **trivially graded**.

4. The free algebra $R = K < X_1, \ldots, X_n >$ may be $\mathbb{Z}$-graded in several different ways. Given any map gr : $\{X_1, \ldots, X_n\} \to \mathbb{Z}$ we obtain a $\mathbb{Z}$-grading on R by putting :
$$R_k = K\text{−span } \left\{ X_{i_1} \ldots X_{i_l}, \sum_{j=1}^{l} \mathrm{gr}(X_{i_j}) = k \right\}$$

5. Put $R = \mathbb{C}[X,Y]/(Y^2 - X^3)$. Then R is $\mathbb{Z}$-graded by putting $\mathrm{gr}(Y) = 3, \mathrm{gr}(X) = 2$; this makes the polynomial $Y^2 - X^3$ homogeneous in $\mathbb{C}[X,Y]$ and then it is clear that the gradation on $\mathbb{C}[X,Y]$ defined by $\deg X = 2, \deg Y = 3$, passes to a gradation on the quotient $\mathbb{C}[X,Y]/(Y^2 - X^3)$ (check !).

6. Let A be a ring G a group acting on A by automorphisms, that is we are given a group morphism $\varphi, G \to \mathrm{Aut} A$. The twisted group ring $A \underset{\varphi}{*} G$ is obtained as the additive group $\oplus_{\sigma \in G} A u_\sigma, \{u_\sigma, \sigma \in G\}$ a set of symbols, with multiplication defined by : $u_\sigma a = \varphi_\sigma(a) u_\sigma$ for all $a \in A, \sigma \in G$, $(a u_\sigma)(b u_\tau) = a \varphi_\sigma(b) u_{\sigma\tau}$ for all $\sigma, \tau \in G$, $a, b \in A$. Put $R = \oplus_{\sigma \in G} A u_\sigma = A \underset{\varphi}{*} G$ for this ring, then $R_\sigma = A u_\sigma$ defines a G-gradation on R. We call R the **skew group ring** (with respect to φ) of G over A.

   In particular, let $A = K$ be a field and Ψ an automorphism of K. Put $K^\Psi = \{x \in K, \Psi(x) = x\}$ and observe that K^Ψ is a subfield of K. Construct the K-vector space $\oplus_{n \in \mathbb{Z}} K X^n$ which is also a K^Ψ-vector space. Define a K^Ψ-algebra structure on $\oplus_{n \in \mathbb{Z}} K X^n$ by extending bilinearly the rule $\lambda_1 X^m \lambda_2 X^n = \lambda_1 \Psi^m(\lambda_2) X^{m+n}$ for $m, n \in \mathbb{Z}$. We denote this K^Ψ-algebra by $K[X, X^{-1}, \Psi]$ and call it the **skew Laurent polynomials** it is also $K \underset{\varphi}{*} \mathbb{Z}$ for $\varphi : \mathbb{Z} \to \mathrm{Aut} K$ given by $\varphi(n) = \Psi^m$ for $n \in \mathbb{Z}$. The positive part of the latter, is denoted by $K[X, \Psi]$, it is called the **skew polynomial ring** in one variable.

For a ring R, R-mod stands for the category of left R-modules. If R is G-graded, then a left R-module M is said to be **G-graded** if there is a family of additive subgroups of M, $\{M_\sigma, \sigma \in G\}$ such that $M = \oplus_{\sigma \in G} M_v$ and $R_\sigma M_\tau \subset M_{\sigma\tau}$ for all $\sigma, \tau \in G$. The elements of $h(M) = \cup_{\sigma \in G} M_\sigma$ are called the homogeneous elements of M. An $m \neq 0$ in M_σ is said to be homogeneous of degree σ. Every nonzero $m \in M$ has a unique decomposition $m = m_{\sigma_1} + \ldots + m_{\sigma_n}$ for some $\sigma_1, \ldots, \sigma_n$ in G, with $m_{\sigma_i} \in M_{\sigma_i}$. A submodule N of M is a **graded submodule** if $N = \oplus_{\sigma \in G}(N \cap M_\sigma)$ or equivalently if for $y \in N$ the homogeneous components y_σ in M, for $\sigma \in G$, are again in N.

3.1.3 Example

Let R be G-graded, M a G-graded R-module and $N \subset M$ an R-submodule. Let N_g be the submodule of M generated by $N \cap h(M)$. It is obvious that N_g is maximal amongst submodules of N which are G-graded submodules of M.

Now consider G-graded R-modules M and N. An R-linear $f : M \to N$ is said to be a **graded morphism of degree** τ if $f(M_\sigma) \subset N_{\sigma\tau}$ for all $\sigma \in G$. The set of graded morphisms of degree τ forms an additive subgroup $\mathrm{HOM}_R(M,N)_\tau$ of $\mathrm{Hom}_R(M,N)$. Also $\mathrm{HOM}_R(M,N) = \oplus_{\tau \in G} \mathrm{HOM}_R(M,N)_\tau$ is a G-graded abelian group ($\mathbb{Z}$-module, where $\mathbb{Z}$ is trivially graded). Composition of a graded morphism, $f : M \to N$, of degree $\sigma \in G$ with a graded morphism

$g : N \to P$ of degree $\tau \in G$, yields a graded morphism $g \circ f$ of degree $\sigma\tau \in G$. If N is a graded submodule of M then the R-module M/N may be G-graded by putting $(M/N)_\sigma = M_\sigma + N/N$, for $\sigma \in G$. The canonical R-linear $\pi : M \to M/N$ is a graded morphism of degree $e \in G$. We consider the category R-gr$_G$ (R-gr if G is understood) of G-graded left R-modules with graded morphisms of degree e, hence $\mathrm{Hom}_{R-\mathrm{gr}}(M, N) = \mathrm{HOM}_R(M, N)_e$ for $M, N \in R$-gr. The category R-gr has direct sums and products. Also if $\{M_\alpha, f_{\alpha,\beta}; \alpha, \beta \in \mathbb{Z}\}$ is an inductive system, resp. projective system, then the R-module $\varinjlim_\alpha M_\alpha$, resp. $\varprojlim_\alpha M_\alpha$, maybe G-graded by putting $(\varinjlim_\alpha M_\alpha)_\sigma = \varinjlim_\alpha (M_\alpha)_\sigma$, resp. $(\varprojlim_\alpha M_\alpha)_\sigma = \varprojlim_\alpha (M_\alpha)_\sigma$, for all $\sigma \in G$.

For the $\mathrm{Hom}_{R-\mathrm{gr}}(M, N)$ both $\mathrm{Ker} f$ and $\mathrm{Coker} f$ are in R-gr. Indeed, since $\mathrm{Im} f$ is G-graded in N it follows that $\mathrm{Coker} f = N/\mathrm{Im} f = \oplus_{\sigma \in G}(N_\sigma + \mathrm{Im} f)/\mathrm{Im} f$ is the cokernel of f in R-gr.

It is straightforward to verify that R-gr is an abelian category satisfying Grothendieck's axioms *Ab*3, *Ab*4, *Ab*3* and *Ab*4* (see for example [S], B. Stenström, 1975). Since also *Ab*5 is easily verified, we may conclude that R-gr is a Grothendieck category. The category of G-graded right R-modules, gr-R, may be defined in the completely symmetric way.

3.1.4 Lemma

Consider $M, N, P \in R$-gr and look at the commutative triangle in R-mod :

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "f"'] & & N \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

where f has degree e. If g, resp. h, has degree e then there exists a graded morphism h', resp. g', of degree e such that $f = g \circ h'$, resp. $f = g' \circ h$.

Proof We deal with the case where $\deg g = e$, the other is similar. Pick $x \in M_\sigma, h(x) = \sum_{\sigma \in G} h(x)_\sigma$ yields that we may define $h' : M \to N, x \mapsto h(x)_\sigma$. The claim follows easily. □

3.1.5 Corollary

A G-graded P is a projective object in R-gr if and only if $\underline{P}$, the R-module underlying P, is a projective R-module.

Proof If P is projective in R-gr then it is clear that $\underline{P}$ is projective in R-mod. So now assume $\underline{P}$ is projective in R-mod. Then there is a surjective map of a gr-free object F to P and this then splits by projectivity of $\underline{P}$. Note that F is gr-free if it has a basis of homogeneous elements and observe that indeed every object of R-gr is a quotient of a gr-free object.

Now we have an exact sequence in R-gr : $F \underset{f}{\longrightarrow} P \longrightarrow 0$ and an R-linear $g : P \to F$ such that $f \circ g = 1_P$. According to the lemma there exists a graded morphism of degree e, $g' : P \to F$ such that $f \circ g' = 1_P$. Thus the exact sequence $F \underset{f}{\longrightarrow} P \longrightarrow 0$ splits in R-gr and therefore P is projective in R-gr. □

A graded left R-submodule L of R is called a **graded left ideal**; if L is two-sided then it is called a **graded ideal of** R.

3.1.6 Lemma

For M, N in R-gr, $\mathrm{HOM}_R(M, N)$ consists of all $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ for which there exists a finite subset of G, say F, such that :

$$(*) \qquad f(M_\sigma) \subset \sum_{\nu \in F} N_{\sigma\nu} \text{ for all } \sigma \in G$$

Proof Suppose $f \in \mathrm{HOM}_R(M, N)$. Then there exists $\sigma_1, \ldots, \sigma_m \in G$ such that $f = f_{\sigma_1} + \ldots + f_{\sigma_m}$ and $f_{\sigma_i} \in \mathrm{HOM}(M, N)_{\sigma_i}, i = 1, \ldots, n$. Clearly f satisfies condition $(*)$. Conversely suppose $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ satisfies $(*)$ for some finite set $F \subset G$.

Look at $x_\sigma \in M_\sigma$; by $(*)$ it follows that : $f(x_\sigma) = \sum_{\nu \in F} y_{\sigma,\sigma\nu}$, for some unique $y_{\sigma,\sigma\nu} \in N_{\sigma\nu}, \nu \in F$. For any $\nu \in F$ we define $f_\nu(x_\sigma) = y_{\sigma,\sigma\nu}$. It is clear that $f_\nu \in \mathrm{HOM}_R(M, N)_\nu$ and $f = \sum_{\nu \in F} f_\nu$. Thus $f \in \mathrm{HOM}_R(M, N)$.

□

3.1.7 Corollary

In each of the following cases we have that $\mathrm{HOM}_R(M, N) = \mathrm{Hom}_R(\underline{M}, \underline{N})$:

a. When the group G is finite.

b. When M is a finitely generated R-module, i.e. $M = Rm_1 + \ldots + Rm_d$ for finitely many $m_1, \ldots, m_d \in M$.

Proof

a. Trivial in view of Lemma 3.1.6.

b. Let $\underline{M}$ be generated by $m_1, \ldots, m_d$ and without loss of generality we may assume that $m_i, i = 1, \ldots, d$, is nonzero homogeneous, say of degree $\alpha_1, \ldots, \alpha_d$. Consider $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ then for $i = 1, \ldots, d$, $f(m_i) = \sum_{j=1}^{t_i} n_{\sigma_{ij}}$ with $n_{\sigma_{ij}} \in N_{\sigma_{ij}}, \sigma_{ij} \in G$.

   For each $i, 1 \leq i \leq d$ we put :

$$F = \bigcup_{i=1}^{d} F_i, F_i = \{\alpha_i^{-1}\sigma_{i1}, \alpha_i^{-1}\sigma_{i2}, \ldots, \alpha_i^{-1}\sigma_{it_i}\}$$

   We claim that $f(M_\sigma) \subset \sum_{\nu \in F} N_{\sigma\nu}$ for all $\sigma \in G$:

   For x_σ in M_σ write : $x_\sigma = \sum_{i=1}^{d} r_i m_i$, with $r_i \in h(R)$, deg $r_i = \sigma\alpha_i^{-1}$. Therefore it follows that :

$$f(x_\sigma) = \sum_{i=1}^{d} r_i f(m_i) = \sum_{i=1}^{d} \sum_{j=1}^{t_i} r_i n_{\sigma_{ij}}$$

   However, deg $r_i n_{\sigma_{ij}} = \sigma\alpha_i^{-1}\sigma_{ij}$, hence $r_i n_{\sigma_{ij}} \in N_{\sigma\nu}$, where $\nu = \alpha_i^{-1}\sigma_{ij} \in F$ for all $1 \leq i \in d$. □

3.1.8 Remark

For $G = \mathbb{Z}$ and M not finitely generated it may happen that $\mathrm{HOM}_R(M, N) \neq \mathrm{Hom}_R(\underline{M}, \underline{N})$. For example let $R = \oplus_{n \in \mathbb{Z}} R_n$ be $\mathbb{Z}$-graded such that $R_n \neq 0$ for all $n \in \mathbb{Z}$. Then there is a $(a_n)_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} R_n$ which is not in $\oplus_{n \in \mathbb{Z}} R_n$. Put $M = R^{(\mathbb{Z})}$ and define $f \in \mathrm{Hom}_R(M, R)$ by putting $f((x_n)_{n \in \mathbb{Z}}) = \sum_{i \in \mathbb{Z}} x_i a_i$. By the lemma clearly $f \notin \mathrm{HOM}_R(M, R)$.

3.1.9 Definition

A G-graded ring R is said to be **strongly graded** if : $R_\sigma R_\tau = R_{\sigma\tau}$ for every $\sigma, \tau \in G$.

In particular when $G = \mathbb{Z}$ then R is strongly graded if and only if $RR_1 = R$, if and only if $R_1 R = R$, if and only if $R_{-1} R_1 = R_0$.

3.1.10 Lemma

A G-graded ring R is strongly graded if and only if $1 \in R_\sigma R_{\sigma^{-1}}$ for every $\sigma \in G$.

Proof If $1 \in R_\sigma R_{\sigma^{-1}}$ for every $\sigma \in G$ then for any $\tau \in G$ it follows that : $R_{\sigma\tau} = 1R_{\sigma\tau} \subset R_\sigma R_{\sigma^{-1}} R_{\sigma\tau} \subset R_\sigma R_\tau$, hence $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$. $\square$

Note that we may strengthen the foregoing lemma to : R is strongly graded if and only if $1 \in R_{\sigma_i} R_{\sigma_i^{-1}}$ for some set of generators $\sigma_i, i \in \mathcal{J}$, of G.

3.1.11 Corollary

If R is strongly graded then R_σ is finitely generated and projective in R_e-mod and in mod-R_e, for every $\sigma \in G$.

Proof From $R_{\sigma^{-1}} R_\sigma = R_e$ it follows that R_σ contains a finitely generated R_e-module L_σ such that $R_{\sigma^{-1}} L_\sigma = R_e$. Then $R_\sigma R_{\sigma^{-1}} L_\sigma = R_\sigma R_e = R_\sigma$ yields $L_\sigma = R_\sigma$ and thus R_σ is finitely generated in R_0-mod; similar from $R_\sigma R_{\sigma^{-1}} = R_e$ the statement about R_σ in mod-R_e follows symmetrically. Write $1 = \sum u_i v_i$ with $u_i \in R_{\sigma^{-1}}, v_i \in R_\sigma$, then right multiplication by u_i defines an R_e-linear $\mu_i : R_\sigma \to R_e$ such that for each $r \in R_\sigma$ we have $r = \sum_i \mu_i(r) v_i$. Therefore by the well-known "Basis Lemma", R_σ is projective in R_e-mod. Similar, a symmetric argumentation leads to projectivity of R_σ in mod-R_e. $\square$

For a G-graded ring R we may define a functor $R \otimes_{R_e} - : R_e\text{-mod} \to R\text{-gr}$, $M \mapsto R \otimes_{R_e} M$ with G-gradation on $R \otimes_{R_e} M$ defined by $(R \otimes_{R_e} M)_\sigma = R_\sigma \otimes_{R_e} M$, for all $\sigma \in G$. A morphism $f \in \mathrm{Hom}_{R_e}(M, N)$ is mapped to $R \otimes_{R_e} f$. On the other hand, we also have the restriction functor : $(-)_e : R\text{-gr} \to R_e\text{-mod}$, $M \mapsto M_e$, where a morphism $f \in \mathrm{Hom}_{R-\mathrm{gr}}(M, N)$ restricts to $f_e : M_e \to N_e$.

A G-graded left module M is said to be strongly graded whenever, for all $\sigma, \tau \in G$ we have $R_\sigma M_\tau = M_{\sigma\tau}$.

3.1.12 Theorem

For a G-graded ring R the following statements are equivalent :

1. R is strongly graded.
2. Every $M \in R$-gr is a strongly graded R-module.
3. The functor $R \otimes_{R_e} -$ and $(-)_e$ define an equivalence between the Grothendieck categories R-gr and R_e-mod.

Proof

1. $\Longrightarrow$ **2.** (in fact 2. $\Longrightarrow$ 1. is obvious). For $\sigma, \tau \in G$ we have $M_{\sigma\tau} = R_e M_{\sigma\tau} = R_\sigma R_{\sigma^{-1}} M_{\sigma\tau} \subset R_\sigma M_\tau$, hence $M_{\sigma\tau} = R_\sigma M_\tau$.

2. $\Longrightarrow$ **3.** (In fact **3.** $\Longrightarrow$ **2.** is easy and left to the reader).

Take $M \in R$-gr and let $\delta_M : R \otimes_{R_e} M_e \to M$ be the canonical R-linear morphism given by : $\delta_M(r \otimes m_e) = rm_e$, for $r \in R$, $m_e \in M_e$. Note that δ_M is graded of degree e. It is easily seen that δ_M is a natural transform of the composition functor $(R \otimes_{R_e} -) \circ (-)_e$ to the identity functor on R-gr. It is clear that δ_M is an epimorphism. If $K_M = \mathrm{Ker}(\delta_M)$, then K_M is a graded submodule of $R \otimes_{R_e} M_e$ and $(K_M)_e = \mathrm{Ker}(\delta_M)_e$ where $(\delta_M)_e : R_e \otimes_{R_e} M_e$ is an isomorphism. Hence $(K_M)_e = 0$ and then also $(K_M)_\sigma = R_\sigma (K_M)_e = 0$, or $K_M = 0$. Conversely, if $M \in R_e$-mod, let $\alpha_M : M \to (R \otimes_R M)_e$ be given by $\alpha_M(x) = 1 \otimes x$. It is clear that α_M defines a natural transform between the identity functor on R_e-mod and the composition functor $(-)_e \circ (R \otimes_{R_e} -)$; (for detail on categories, functors, equivalences and natural transforms, cf. [15]). $\square$

3.1.13 Corollary

If R is strongly G-graded and M, N are graded R-modules, $f : M \to N$ in $\mathrm{Hom}_{\mathrm{gr}}(M, N)$. Then f is a monomorphism, resp. epimorphism, resp. isomorphism, if and only if $f_\sigma : M_\sigma \to N_\sigma$, is resp. a monomorphism, epimorphism, isomorphism, for some $\sigma \in G$ where $f_\sigma \in \mathrm{Hom}_{R_e}(M_\sigma, N_\sigma)$.

Proof We have a functor $T_\sigma : R\text{-gr} \to R\text{-gr}$ defined by $T_\sigma(M)$ has underlying R-module $\underline{M}$ but gradation is defined by $T_\sigma(M)_\tau = M_{\tau\sigma}$ for every $\sigma \in G, \tau \in G$. This functor, for every $\sigma \in G$ defines an equivalence of R-gr with itself. Hence f is monomorphic, epimorphic, isomorphic if and only if $T(\sigma)f : T_\sigma(M) \to T_\sigma(N)$ is resp. monomorphic, epimorphic, isomorphic. Now $T_\sigma(M)_e = M_\sigma, T_\sigma(N)_e = N_\sigma$, hence it follows from the foregoing theorem that the statements of the corollary hold. $\square$

Recall that a bimodule ${}_R M_R$ is **invertible** if there exists an R-bimodule ${}_R N_R$ such that $M \otimes_R N \cong R \cong N \otimes_R M$ as R-bimodules.

3.1.14 Proposition

For a strongly G-graded ring R :

1. For $M \in R$-gr and all $\sigma, \tau \in G$, the canonical morphism $R \otimes_{R_e} M_\sigma \to T_\sigma(M), r \otimes x \mapsto rx$, is an isomorphism.

2. For every $\sigma, \tau \in G$ the canonical morphism :$R_\sigma \otimes_{R_e} R_\tau \to R_{\sigma\tau}$ is an isomorphism of R_e-bimodules.

3. For every $\sigma \in G, R_\sigma$ is an invertible R_e-bimodule.

Proof

1. Immediately from Theorem 3.1.12.

2. By the first part, $R \otimes_{R_e} R_\tau \mapsto T_\tau(R)$ is an isomorphism, hence $(R \otimes_{R_e} R_\tau)_\sigma = T_\tau(R)_\sigma = R_{\sigma\tau}$, therefore $R_\sigma \otimes_{R_e} R_\tau \cong R_{\sigma\tau}$ for $\sigma, \tau \in G$.

3. Follows from 2. □

3.1.15 Corollary

Let R be strongly G-graded, then :

a. If $M \in R$-gr, then $M = 0$ if and only if $M_\sigma = 0$ for some $\sigma \in G$.

b. Every graded left ideal L of R is generated by $L_e, L = RL_e$.

c. If $\varphi : R \to S$ is a graded ring morphism of degree e between strongly graded rings R and S, then φ is injective, resp. surjective bijective if and only if $\varphi_e : R_e \to S_e$ is injective, resp. surjective, bijective.

Proof

a. $M \in R$-gr is strongly graded, hence $M = RM_\sigma$ for every $\sigma \in G$.

b. L is strongly graded, thus $L_\sigma = R_\sigma L_e$ and $L = RL_e$ follows.

c. If φ_e is injective, then $\mathrm{Ker}\varphi$ is graded and $(\mathrm{Ker}\varphi)_e = \mathrm{Ker}\varphi_e$ implies $\mathrm{Ker}\varphi = 0$ in view of **b.**. If φ_e is surjective then $\varphi(R_{\sigma^{-1}}) \subset S_{\sigma^{-1}}$ and $\varphi(R_e) = S_e$ entails : $S_e = \varphi(R_{\sigma^{-1}})\varphi(R_\sigma) \subset S_{\sigma^{-1}}\varphi(R_\sigma)$. Therefore we have $S_{\sigma^{-1}}\varphi(R_\sigma) = S_e$ and thus $S_\sigma = S_\sigma S_e = S_\sigma S_{\sigma^{-1}}\varphi(R_\sigma) = \varphi(R_\sigma)$ for every $\sigma \in G$. If φ_e is bijective then φ is injective and surjective thus bijective. The other implications are obvious. □

For a detailed treatment of graded ring theory we refer to [16], [17].

3.2 Filtered Rings and Modules

We may define filtered rings with respct to any totally ordered group but here we will restrict attention to $\mathbb{Z}$-filtrations.

3.2.1 Definition

A **filtration** FR on a ring R is an ascending chain :

$$\ldots \subset F_n R \subset F_{n+1} R \subset \ldots \subset R$$

consising of additive subgroups of R such that for all $n, m \in \mathbb{Z}$, $F_n R F_m R \subset F_{n+m} R$ and moreover $1 \in F_0 R$ (unlike what happens for $\mathbb{Z}$-gradation this does not follow now from the first condition). A K**-algebra** A **is filtered** if A is filtered and $K \subset F_0 A$. A filtration FR of a ring R is **exhaustive** if $\cup_n F_n R = R$ and FR is **separated** if $\cap_n F_n R = 0$. From hereon we only consider exhaustive filtrations.

3.2.2 Examples

1. Let R be a ring with filtration FR. Any surjective $\varphi : R \to S$ (ring homomorphism) defines $F_n S = \varphi(F_n R)$, a filtration on S. If FR is not separated then $\cap_{n \in \mathbb{Z}} F_n R = I$ is an ideal and the filtration induced by FR on R/I is separated.

2. Any $\mathbb{Z}$-graded ring is also filtered by the gradation-filtration, $F_p R = \oplus_{n \leq p} R_n$.

3. As in **2.** we may define a filtration on the free algebra $R = K < \mathcal{X} >$, for any set $\mathcal{X}$, by putting $\mathcal{X} \subset R_1$. As every algebra is an epimorphic image of some $K < \mathcal{X} >$, the filtration on $K < \mathcal{X} >$ induces a filtration on an arbitrary K-algebra A via $K < \mathcal{X} > \twoheadrightarrow A$. We have $A = K < a_i, i \in J >$, $F_0 A = K, F_1 A = \oplus_{i \in J} K a_i$, $F_n A = (F_1 A)^n$ for $n > 0$; such a filtration is called a **standard filtration**.

3.2.3 Definition

An R-module M is a **filtered module** if there is an ascending chain of additive subgroups : $\ldots \subset F_n M \subset F_{n+1} M \subset \ldots$, such that for all $n, m \in \mathbb{Z}$ we have $F_n R F_m M \subset F_{n+m} M$. We will restrict attention to exhaustive filtrations i.e. we assume that $M = \cup_n F_n M$.

It is obvious that $F_n M$ is a left $F_0 R$-module, $F_n R$ is an $F_0 R$-bimodule.

To a filtration FM on an R-module M we may associate a topology on M by considering FM as a filter of neighborhoods of 0. For $x \in M$ a filter of neigbourhoods will then be given by $\mathcal{V}(x) = \{x + F_n M, n \in \mathbb{Z}\}$.

3.2.4 Proposition

1. The sum and scalar multiplication of M are continuous in the topology induced by FM.

2. If FM is separated then the induced topology is Hausdorff, moreover the induced topology is metric.

Proof

1. The map $m+ : M \to M, x \mapsto m + x$ maps a filter of neighbourhoods to a filter of neighbourhoods, the inverse $(-m)+$ is also of that type hence $m+$ is a homeomorphism. It follows that the operation $+$ is continuous.

   In order to establish continuity of λ, for $\lambda \in R$ we use the exhaustivity of FR to select $k \in \mathbb{Z}$ such that $\lambda \in F_k R$. If $\lambda y \in x + F_n M$ then $\lambda(y + F_{n-k}M) \subset x + F_n M$. Hence the inverse image of $x + F_n M$ for the map λ is open, thus λ is continuous (but not necessarily a homeomorphism). If FM is separated and $x \neq y$ in M then there is an n such that $x - y \notin F_n M$, therefore : $(x + F_n M) \cap (y + F_n M) = \emptyset$ and this yields the Hausdorff property for the topology. If $y \neq x$ define : $d(x, y) = e^k$ where k is minimal such that $x - y \in F_k M$ (e the basis of the natural logarithm). From $d(x, y) = \max(d(x, z), d(y, z) \leq d(x, z) + d(y, z)$, for $z \in M$. The open balls for the metric given by $d(-, -)$ are exactly the elements of the filter of neighbourhoods for the filtration topology. □

Observe that the proposition above also holds for $M = R$ where scalar multiplication is replaced by the multiplication of the ring, hence FR defines a structure of topological ring on R.

3.2.5 Definition

Let I be an ideal of R. The **I-adic filtration** of R is given by :

$$\ldots I^n \subset I^{n-1} \subset \ldots \subset R \subset R \subset R \ldots$$

(so we view this as an ascending filtration !)

3.2.6 Example

Look at $I = (X)$ in $K[X]$. The I-adic filtration defines the I-adic topology on $K[X]$, the latter is a metric topology so we can look at the completion $K[X]^\wedge$ of $K[X]$ in this topology.

Let us recall the construction of the completion of a filtered ring R. Let $\mathcal{R}$ be the set of Cauchy sequences $(r_i)_{i \in \mathbb{N}}$, these are sequences with $r_i \in R$ such that for all $\varepsilon \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for all $i, j > N$ we have that $d(r_i, r_j) < \varepsilon$. On $\mathcal{R}$ we define an equivalence relation $(r_i)_{i \in \mathbb{N}} \sim (s_i)_{i \in \mathbb{N}}$ if and only if $(r_i - s_i)_{i \in \mathbb{N}}$ converges to zero. The completion of R with respect to FR is now $\mathcal{R}/\sim$ together with the componentwise addition and multiplication (these are well defined on $\mathcal{R}/\sim$ because $+$ and . are continuous). We write $\widehat{R}$ for the completion at FR and there is an injective ring morphism (if FR is separated), $R \to \widehat{R}$ mapping $r \in R$ to the class of the constant sequence $(r_i)_{i \in \mathbb{N}}$ with $r_i = r$ for all i. In case we look at the X-adic topology on $K[X]$ the completion is denoted by $K[[X]]$. By construction every sequence of polynomials of type :

$$(a_0, a_0 + a_1 X, a_0 + a_1 X + a_2 X^2, a_0 + a_1 X + a_2 X^2 + a_3 X^3, \ldots)$$

is in $\mathcal{R}$ and any two different such sequences are not equivalent. In fact every sequence in $\mathcal{R}$ is equivalent to a unique sequence of the foregoing type. For e^{-n} we can find an $N \in \mathbb{N}$ such that $d(r_i, r_j) < e^{-n}$ for $i, j > N$, in other words $r_j \in r_i + (X^n)$. In $r_i + (X^n)$ there is a unique element of the form $a_0 + \ldots + a_{n-1}X^{n-1}$ independent of the choice of $i > N$; call this element

r'_{n-1}. The sequence $(r'_i)_{i\in\mathbb{N}}$ thus constructed is of the above type and it is obviously equivalent to $(r_i)_{i\in\mathbb{N}}$.

It follows that we may look at $K[[X]]$ as the ring of formal power series. We easily verify that $K[[X]]$ inherits the filtration from $FK[X]$ by putting :

$$F_nK[[X]] = \{a_nX^n + a_{n+1}X^{n+1} + \ldots, a_i \in K\}$$

3.2.7 **Example.** The p-adic numbers

Take $R = \mathbb{Z}, I = (p) = p\mathbb{Z}$ for a prime number p of $\mathbb{Z}$. The I-adic filtration defines the p-adic topology on $\mathbb{Z}$. The completion of $\mathbb{Z}$ with respect to the p-adic filtration yields $\mathbb{Z}_{(p)}$, the ring of p-adic numbers, which may be viewed as formal power series in p. The p-adic filtration on $\mathbb{Z}$ can be extended to a filtration of $\mathbb{Q}$; look at the localization $\mathbb{Z}_p = \{a/b, p$ does not divide $b\} \subset \mathbb{Q}$ and define $F_p\mathbb{Q}$ by : $\ldots \subset p^2\mathbb{Z}_p \subset p\mathbb{Z}_p \subset \mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset \ldots \subset \mathbb{Q}$.

A filtration FM is said to be **left-limited or discrete** if there is an $n_0 \in \mathbb{Z}$ such that for all $i < n_0, F_iM = 0$ (the topology on M associated to FM is then the discrete topology). A special discrete topology is a **positive filtration** where n_0 in the foregoing is at least 0.

3.2.8 Examples

1. Let $\varphi : A \to A$ be an injective ring morphism and $\delta : A \to A$ a φ-derivation, i.e. for $x, y \in A$, $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$. The ring of **skew polynomials** is obtained by adjoining a varable t to A and defining multiplication as follows : $ta = \varphi(a)t + \delta(a)$. We write $R = A[t, \varphi, \delta]$ for this ring. Obviously A is subring of R. The fact that φ is injective allows to define a degree function deg as the degree in t and we arrive at the degree filtration FR by putting $F_nR = \{f(t) \in A[t, \varphi, \delta], \deg f(t) \leq n\}$, $F_0R = A$.

2. Let g be a Lie algebra over K and $U(g)$ the universal $(K-)$ enveloping algebra of g (see Chapter II). Since $U(g)$ is a quotient of the free algebra on $\{x_1, \ldots, x_d\}$ which is any K-basis of g the gradation filtration of $K < x_1, \ldots, x_d >$ induces a positive filtration $FU(g)$ with $F_0U(g) = K$, $F_1U(g) = K \oplus Kx_1 \oplus \ldots \oplus Kx_d$, $F_nU(g) = (F_1U(g))^n$ for $n > 1$.

3. Let A be a commutative k-algebra and $\mathcal{D}$ any A-submodule of Der_kA. Let $A[\mathcal{D}]$ be the k-algebra generated by A and $\mathcal{D}$ (in End_kA). Then $A[\mathcal{D}]$ has a positive (standard) filtration corresponding to $\mathcal{D}$ as a generating set over A, i.e. $F_0A[\mathcal{D}] = A$, $F_1A[\mathcal{D}] = A + \sum_{\delta\in D} A\delta$, $F_nA[\mathcal{D}] = (F_1A[\mathcal{D}])^n$ for $n > 1$. In particular for $\mathcal{D} = \mathrm{Der}_kA$ we obtain the derivation ring $\Delta(A) = A[\mathrm{Der}_kA]$ with its standard filtration defined by a k-basis of Der_kA. In particular we may take $A = k[x_1, \ldots, x_n]$ and then we obtain the derivation ring $\Delta(A) = \mathbb{A}_n(k)$ (see Chapter I), the n-th Weyl algebra. The standard filtration on $\mathbb{A}_n(k)$ may be defined by putting $x_1 \ldots, x_n$ and $y_1, \ldots, y_n$ in $F_1\mathbb{A}_n(k), F_0\mathbb{A}_n(k) = k$ and $F_m\mathbb{A}_n(k) = (F_1\mathbb{A}_n(k))^m$, for $m > 0$. We call this filtration on $\mathbb{A}_n(k)$ the **Bernstein filtration** on $\mathbb{A}_n(k)$, thus $F_m\mathbb{A}_n(k)$ is the k-space generated by all $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}y_1^{\beta_n}, \ldots, y_n^{\beta_1}$ with $\alpha_1 + \ldots + \alpha_n + \beta_1 + \ldots + \beta_n \leq m$ and it is a positive filtration.

   We have seen that $\mathbb{A}_1(k) = k[x][y, \frac{\partial}{\partial x}]$, hence in Example 3.2.3(1) with $\varphi = 1_A$ and $A = k[x]$ we obtain another filtration on $\mathbb{A}_1(k)$ called the Σ-filtration say $F'\mathbb{A}_1(k)$, with

$F'_0\mathbb{A}_1(k) = k[x]$ and $F'_m\mathbb{A},(k) = \{f(y) \in A[y, \frac{\partial}{\partial x}], \deg_y f(y) \leq m\}$. Sometimes the Σ-filtration is also called the operator filtration. It may in the obvious way be defined for $\mathbb{A}_n(k)$ over $k[x_1, \ldots, x_n]$.

4. Let k be a field, A a commutative k-algebra and M, N left A-modules. Give $\mathrm{Hom}_k(M, N)$ the structure of an $A \otimes_k A$-module by defining $((a \otimes b)\theta)(m) = a\theta(bm)$ for $a, b \in A, \theta \in \mathrm{Hom}_k(M, N)$ and $m \in M$. Write $\mu : A \otimes_k A \to A$ for the multiplication map $\mu(a \otimes b) = ab$ for a, b in A and put $J = \mathrm{Ker}\mu$.

   Define the space of **k-linear differential operators from M to N of order at most n** by : $\mathcal{D}^n_A(M, N) = \{\theta \in \mathrm{Hom}_k(M, N), J^{n+1}\theta = 0\}$, where $J^0 = A \otimes_k A$. Put $\mathcal{D}_A(M, N) = \cup_{n=0}^{\infty} \mathcal{D}^n_A(M, N)$. Note that J is generated by $\{1 \otimes a - a \otimes 1, a \in A\}$ (verify this).

   If we write $[\theta, a] = (1 \otimes a - a \otimes 1)\theta = \theta a - a\theta$, for $a \in A$ and $\theta \in \mathrm{Hom}_k(M, N)$, then $\mathcal{D}^0_A(M, N) = \mathrm{Hom}_A(M, N)$ and $D_A(M, N) = 0$ if and only if $\mathrm{Hom}_A(M, N) = 0$. We may alternatively define $\mathcal{D}^n_A(M, N)$ inductively by $\mathcal{D}^{-1}_A(M, N) = 0$ and for $n \geq 0$, $\mathcal{D}^n_A(M, N)$ $= \{\theta \in \mathrm{Hom}_k(M, N), [\theta, a] \in \mathcal{D}^{n-1}_A(M, N)$ for all $a \in A\}$. It follows that $\mathcal{D}^n_A(M, N) \subset \mathcal{D}^{n+1}_A(M, N)$ for $n \geq 0$. Let us write $\mathcal{D}_A(A, A) = \mathcal{D}(A)$. Then for f and g of $\mathcal{D}(A)$ we have $fg \in \mathcal{D}(A)$. Indeed, if f has order p and g has order q, then fg has order at most $p + q$. Thus $\mathcal{D}^n_A(A)\mathcal{D}^m_A(A) \subset \mathcal{D}^{n+m}_A(A)$ and thus $\mathcal{D}(A)$ is a filtered k-algebra with filtration $F\mathcal{D}(A)$ given by $F_n\mathcal{D}(A) = \mathcal{D}^n_A(A), n \in \mathbb{N}$. $\mathcal{D}(A)$ is called the **ring of differential operators of A**. We refer to [Bj] for the detailed theory of differential operators .

   The category R-filt for a filtered ring R consists of the filtered R-modules with filtered morphisms, where an R-linear $f : M \to N$, with M and N filtered R-modules, is a **filtered morphism** if $f(F_nM) \subset F_nN$.

   An R-submodule M' of a filtered R-modules M is called a **filtered submodule** if $F_nM' \subset F_nM$ for all $n \in \mathbb{Z}$. Any R-submodule M' defines a filtered submodule of M if we put on M' the induced filtration : $F_nM' = F_nM \cap M'$, for all $n \in \mathbb{Z}$. For an R-submodule H of M the **quotient filtration** on $M/H = Q$ is defined by putting $F_nQ = F_nM + H/H$, for $n \in \mathbb{Z}$. In the category R-filt there exist direct sums, products, inductive limits, $\varinjlim$, and projective limits, $\varprojlim$, where resp.

$$F_p \varinjlim_i M_i = \varinjlim_i F_pM_i, F_p(\varprojlim_i M_i) = \varprojlim_i F_pM_i$$

   Direct sums and inductive limits of exhaustive filtrations are again exhaustive. This property fails for the product (hence for projective limits). For M, N in R-filt and $f : M \to N$ a filtered morphism, we say that f has the **Artin-Rees-property** if there exists a $c \in \mathbb{Z}$ such that for all $n \in \mathbb{Z} : \mathrm{Imf} \cap F_nN \subset f(F_{n+c}M)$. We say that f is **strict** if for all $n \in \mathbb{Z} : f(F_nM) = \mathrm{Imf} \cap F_nN$. Every strict morphism has the Artin Rees property.

   For example if N is an R submodule of a filtered module M then the inclusion $N \hookrightarrow M$ becomes a strict morphism when FN is the filtration of N induced from FM; similarly M/N endowed with the quotient filtration makes the canonical R-linear $M \to M/N$ into a strict filtered morphism.

Let f be a filtered morphism $M \to N$ and consider the canonical (F_0R-linear) $\varphi_M^n : M \to M/F_nM$, $\varphi_N^n : N \to N/F_nN$, for every $n \in \mathbb{Z}$. Since $f(F_nM) \subset F_nN$ for all $n \in \mathbb{Z}$, we obtain a commutative diagram :

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi_M^n"'] & N \arrow[d, "\varphi_N^n"] \\ M/F_nM \arrow[r, "f_n"'] & N/F_nN \end{tikzcd}$$

3.2.9 Lemma

The filtered morphism $f : M \to N$ is strict if and only if the associated map : $\mathrm{Ker}f \to \mathrm{Ker}f_n$ is an epimorphism for all $n \in \mathbb{Z}$.

Proof Suppose f is strict. If $x \in \mathrm{Ker}f_n$ then $x = \varphi_M^n(e)$ for $e \in M$ and $\varphi_N^n(f(e)) = 0$, i.e. $f(e) \in f(M) \cap F_nN$. Hence there is an $x_n \in F_nM$ such that $f(e) = f(x_n)$, i.e. $e - x_n \in \mathrm{Ker}f$ and $\varphi_M^n(e - x_n) = \varphi_M^n(e) = x$, hence $\mathrm{Ker}f \to \mathrm{Ker}f_n$ is surjective. Conversely assume $\mathrm{Ker}f \to \mathrm{Ker}f_n$ is surjective for all $n \in \mathbb{Z}$. Pick $x' \in f(M) \cap F_nN$. Then $x' = f(x)$ for some $x \in M$. Now : $f^n\varphi_M^n(x) = \varphi_N^n(f(x)) = \varphi_N^n(x') = 0$ because $x' \in F_nN$. Thus $\varphi_M^n(x) \in \mathrm{Ker}f_n$. By the assumption $\varphi_M^n(x) = \varphi_M^n(y)$ with $y \in \mathrm{Ker}f$, it follows that $x - y \in \mathrm{Ker}\varphi_M^n = F_nM$ and $f(x) = f(x - y)$. Thus $x' = f(x) \in f(F_nM)$ and the proof is finished. $\square$

3.2.10 Corollary

If $f : M \to M$ is a filtered morphism such that $f^2 = f$, then f is strict.

Now we start from a ring R with a **separated filtration** FR. For $n \in \mathbb{Z}$, put $G_n(R) = F_nR/F_{n-1}R$ and $G(R) = \oplus_{n\in\mathbb{Z}} G(R)_n$ as additive groups. For $a_n \in F_nR$ write $\overline{a}_n = a_n \mathrm{mod} F_{n-1}R$, and define multiplication by : $\overline{a}_n.\overline{a}_m = (a_n a_m)\mathrm{mod} F_{n+m-1}R$.

3.2.11 Lemma

The multiplication on $G(R)$ is well defined and this makes $G(R)$ into a $\mathbb{Z}$-graded ring.

Proof For $f_{n-1} \in F_{n-1}, f_{m-1} \in F_{m-1}$ we obtain :

$$\begin{aligned} &(a_n + f_{n-1})(a_m + f_{m-1})\mathrm{mod}F_{n+m-1}R = \\ &\quad = (a_n a_m + f_{n-1}a_m + a_n f_{m-1} + f_{n-1}f_{m-1}) \bmod F_{n+m-1}R = \\ &\quad = (a_n a_m) \bmod F_{n+m-1}R \end{aligned}$$

Hence the multiplication is defined independent of the chosen representatives for $\overline{a}_n, \overline{a}_m$. That $G(R)$ is now a $\mathbb{Z}$-graded ring is easily verified. $\square$

We call $G(R) = G_F(R)$ the **associated graded ring** of R with respect to FR.

3.2.12 Remarks

1. If R is a $\mathbb{Z}$-graded ring and we consider the grading filtration $F^g R, F_n^g R = \oplus_{m \leq n} R_m$ for $n \in \mathbb{Z}$. Then $G(R)_n = \oplus_{m \leq n} R_m / \oplus_{m \leq n-1} R_m = R_n$ for all $n \in \mathbb{Z}$ and one easily verifies that : $G_g(R) \cong R$.

2. Let M be a filtered R-module with (separated) filtration FM. Define additive subgroups $G_n(M) = F_n M / F_{n-1} M$. Now define a $G(R)$ multiplication on $G(M) = \oplus_{n \in \mathbb{Z}} G_n(M)$, as follows : $\overline{a}.\overline{m} = \overline{am}$, for $\overline{a} = a \ \mathrm{mod} F_{p-1} R$, $m \in F_\delta M$, $\overline{m} = m \mathrm{mod} F_{\delta-1} M$, where $\overline{am} = (am) \mathrm{mod} F_{p+\delta-1} M$ because $am \in F_{p+\delta} M$.

   Since FR is separated, there is for every $x \neq 0$ in R an n such that $x \in F_n R - F_{n-1} R$. We define the **principal symbol map** $\sigma : R \to G(R), \sigma(0) = 0$, and $\sigma(x) = x \mathrm{mod} F_{n-1} R$ if n is associated to x as above. For $ab \neq 0$ we have $\sigma(ab) \neq 0$ but $\sigma(a)\sigma(b) = 0$ is possible.

3.2.13 Property

If $\sigma(a)\sigma(b) \neq 0$ then $\sigma(a)\sigma(b) = \sigma(ab)$.

Proof Take $a \in F_n R - F_{n-1} R, b \in F_m M - F_{m-1} R$, then $ab \in F_{n+m} R$. Since $0 \neq \sigma(a)\sigma(b) = (ab) \mathrm{mod} F_{n+m-1} R$, we have $ab \notin F_{n+m-1} R$, thus $ab \in F_{n+m} R - F_{n+m-1} R$. Hence $\sigma(ab) \in G_{n+m}(R)$ and from the foregoing it then follows $\sigma(a)\sigma(b) = \sigma(ab)$. □

3.2.14 Proposition

1. If $\sigma(b)$ is a non-zerodivisor in $G(R)$ then $\sigma(a)\sigma(b) = \sigma(ab)$ for all $a \in R$.

2. In case $G(R)$ is a domain then so is R.

Proof

1. If $\sigma(b)$ is not a zerodivisor then $\sigma(a) \neq 0$ yields $\sigma(a)\sigma(b) \neq 0$, hence $\sigma(a)\sigma(b) = \sigma(ab)$.

2. If $G(R)$ is a domain, then $\sigma(ab) = \sigma(a).\sigma(b) \neq 0$ and $\sigma(ab) \neq 0$ yields $ab \neq 0$.

3.2.15 Property

Take $a \in F_n R - F_{n-1} R$, $b \in F_m R - F_{m-1} R$, then :

$$
\begin{aligned}
\sigma(a+b) &= \sigma(a) \text{ if } n > m \\
&= \sigma(b) \text{ if } m > n \\
&= \sigma(a) + \sigma(b) \text{ if } m = n \text{ and } \sigma(a) + \sigma(b) \neq 0 \\
&\in \oplus_{i<n} G_i(R) \text{ if } m = n \text{ and } \sigma(a) + \sigma(b) = 0
\end{aligned}
$$

Proof If $n > m$, then $a + b \in F_nR - F_{n-1}R$ and $\sigma(a + b) = (a + b) \bmod F_{n-1}R = a \bmod F_{n-1}(R) = \sigma(a)$. Similar for $m > n$. In case $m = n$ then $\sigma(a) + \sigma(b) = 0$ if and only if $a + b \in F_{n-1}R$; then $a + b \notin F_{n-1}R$ yields $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $a + b \in F_{n-1}R$ then yields $\sigma(a + b) \in G_i(R)$ for some $i < n$. □

We now describe the associated graded ring of filtered rings given by generators and relations.

Consider a finitely generated K-algebra A given as $K < X_1, \ldots, X_n > /(\mathcal{R})$. We put on $K < X_1, \ldots, X_n >$ the standard filtration which is also the gradation filtration. This filtration is transferred to $A = K < X_1, \ldots, X_n > /(\mathcal{R})$ by the standard projection $\pi = K < X_1, \ldots, X_n > \to A$. The associated graded ring of $K < X_1, \ldots, X_n >$ is isomorphic to $K < X_1, \ldots, X_n >$; let $\sigma : K < X_1, \ldots, X_n > \to K < X_1, \ldots, X_n >$ be the principal symbol map. The associated graded ring of A with respect to FA is denoted by $G(A)$ and the principal symbol for FA is denoted by $\overline{\sigma} : A \to G(A)$.

3.2.16 Proposition

The associated graded ring of $A = K < X_1, \ldots, X_m > /(\mathcal{R})$ is

$$G(A) = K < X_1, \ldots, X_m > /(\sigma(R), R \in \mathcal{R})$$

Proof Define a linear graded map $\overline{\pi} : K < X_1, \ldots, X_n > \to G(A)$ as follows. For f homogeneous in $K < X_1, \ldots, X_n >$ put $f \bmod F_mK < X_1, \ldots, X_n > \mapsto \pi(f) \bmod F_mA$, if $f \in F_{m+1}K < X_1, \ldots, X_n >$, extended linearly. This map is well defined (as one easily checks) and it is an algebra morphism because :

$$\overline{\pi}(f \bmod F_mK < X_1, \ldots, X_n > g \bmod F_kK < X_1, \ldots, X_n >) = \\ = \overline{\pi}(fg \bmod F_{m+k+1}K < X_1, \ldots, X_n >)$$

We have that $\overline{\sigma} \circ \pi = \overline{\pi} \circ \sigma$, hence the following diagram is commutative

$$\begin{tikzcd} K < X_1, \ldots, X_n > \arrow[r, "\sigma"] \arrow[d, "\pi"] & K < X_1, \ldots, X_n > \arrow[d, "\overline{\pi}"] \\ A \arrow[r, "\overline{\sigma}"'] & G(A) \end{tikzcd}$$

because every homogeneous element of $K < X_1, \ldots, X_n >$ is of the form $\sigma(h)$ for h in $K < X_1, \ldots, X_n >$. What is $\mathrm{Ker}\overline{\pi}$? Look at a homogenous $f \in \mathrm{Ker}\overline{\pi}$ of degree $m \in \mathbb{Z}$. Then $f = \sigma(h)$ for some h in $K < X_1, \ldots, X_n >$ such that $\overline{\pi}\sigma(f) = 0$ hence $\overline{\sigma}\pi(h) = 0$. Since $\overline{\sigma}(a) = 0$ if and only if $a = 0$ (as FA is separated) we must have $h \in \mathrm{Ker}\pi$. Since $\mathrm{Ker}\overline{\pi}$ is graded it is the ideal generated by its homogeneous elements, hence $\mathrm{Ker}\overline{\pi} = (\sigma(h), h \in \mathrm{Ker}\pi) = (\sigma(R), R \in \mathcal{R})$. □

3.2.17 Corollary

Let $\mathbb{A}_n(K)$ be the $n^{-\mathrm{th}}$ Weyl algebra with the Bernstein filtration i.e.

$$\mathbb{A}_n(K) = K < X_1, \ldots, X_n, Y_1, \ldots, Y_n > /(Y_1X_1 - X_1Y_1 - 1, \ldots, Y_nX_n - X_nY_n - 1, \\ X_iX_j - X_jX_i, Y_iY_j - Y_jY_i, Y_iX_j - X_jY_i \text{ for } i \neq j)$$

Then $G\mathbb{A}_n(K) = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, because the foregoing yields :

$$G\mathbb{A}_n(K) = K < X_1, \ldots, X_n, Y_1, \ldots, Y_n > / \\ (X_iX_j - X_jX_i, Y_iY_j - Y_jY_i, Y_iX_j - Y_jY_i)$$

Hence the associated graded algebra for the Bernstein filtration on the Weyl algebra is the polynomial algebra.

3.2.18 Theorem

The linear map $\iota : \mathbb{C}[X, Y] \to \mathbb{A}_1(\mathbb{C})$, $X^iY^j \mapsto x^iy^j$, is a bijection. In other words, every element z of $\mathbb{A}_1(\mathbb{C})$ may in a unique way be written as : $p_n(x)y^n + \ldots + p_s(x)y + p_o(x)$, with $p_i(x) \in \mathbb{C}[x]$, $i = 0, \ldots, n$.

Proof Consider the Bernstein filtration on $\mathbb{A}_1(\mathbb{C})$ with principal symbol map σ. We have obtained before that $G(\mathbb{A}_1(\mathbb{C})) = \mathbb{C}[X, Y]$ with $\sigma(x) = X, \sigma(y) = Y$ because $x, y \in F_1\mathbb{A}_1(\mathbb{C}) - \mathbb{C}$. Hence $\sigma\iota(X^iY^j) = X^iY^j$ and we obtain that $\sigma\iota f(X, Y)$ is the term of highest degree in $f(X, Y)$. Now we define $\overline{\sigma} : \mathbb{A}_1(\mathbb{C}) \to \mathbb{C}[X, Y]$ by putting $\overline{\sigma}(o) = 0$ and $\overline{\sigma}(a) = \sigma(a) + \overline{\sigma}(a - \iota\sigma(a))$ inductively, i.e. if $a \in F_n\mathbb{A}, (\mathbb{C})$ then $a - \iota\sigma(a) \in F_{n-1}\mathbb{A}_1(\mathbb{C})$. Hence for $a \subset F_0\mathbb{A}_1(\mathbb{C}) = \mathbb{C}$ we have $\overline{\sigma}(a) = \sigma(a) = a$ etc...
For example :

$$\begin{aligned} \overline{\sigma}(xyx) &= X^2Y + \overline{\sigma}(xyx - x^2y) = \\ &= X^2Y + \overline{\sigma}(x(yx - xy)) = X^2Y + \overline{\sigma}(-x) \\ &= X^2Y - X \end{aligned}$$

Using the inductive definition we may also verify that $\iota\overline{\sigma} = I_{\mathbb{A}_1(\mathbb{C})}, \overline{\sigma}\iota = I_{\mathbb{C}[X,Y]}$; that is, use induction on the degree of $f \in \mathbb{C}[X, Y]$ together with the fact that $f(X, Y) - \sigma\iota f(X, Y)$ has lower filtration degree than $f(X, Y)$. It follows that ι is a bijection ! $\square$

For a filtered module M we let $G_n(M) = F_nM/F_{n-1}M$, assuming that FM is a separated filtration on M. Then $G(M) = \oplus_{n\in\mathbb{Z}}G_n(M)$. If $\overline{m}_n \in G_n(M)$, let $m_n \in F_nM - F_{n-1}M$ represent $\overline{m}_n$, for $\overline{a}_m \in G_m(R)$ we define $\overline{a}_m.\overline{m}_n$ as $a_mm_n \mathrm{mod} F_{n+m-1}M$. It is not hard to verify that $G(M)$ is a $\mathbb{Z}$-graded $G(R)$-module; we call $G(M)$ the **associated graded module of** M (with respect to FM).

If $\mathrm{Hom}_{FR}(M, N)$ stands for the group of filtered R-module morphisms then any $f \in \mathrm{Hom}_{FR}(M, N)$ induces canonical maps $f_n : F_nM/F_{n-1}M \to F_nN/F_{n-1}N$. It is clear that $G(f) = \oplus_{n\in\mathbb{Z}}f_n$ defines a graded morphism of degree zero from $G(M)$ to $G(N)$, i.e. $G(f) \in \mathrm{Hom}_{G(R)-\mathrm{gr}}(G(M), G(N))$. If $g \in \mathrm{Hom}_{FR}(N, K)$ then $G(g \circ f) = G(g) \circ G(f)$, moreover if I_M is the identity morphism on $\mathrm{Hom}_{FR}(M, M)$ then $G(I_M) = I_{G(M)}$. Then G defines a functor $G : R\text{-filt} \to G(R)\text{-gr}$.

3.2.19 Proposition

With notation as before we have :

i) If $M \in R$-filt has separated filtration FM then $G(M) = 0$ if and only if $M = 0$.

ii) If $M \in R$-filt has discrete filtration ($F_pM = 0$ for $p < n_0$, some $n_0 \in \mathbb{Z}$) then $G(M)$ is left limited in the sense that $G_i(M) = 0$ for all $i < m_0$, some $m_0 \in \mathbb{Z}$.

iii) The functor G commutes with filtered direct sums, filtered products and filtered inductive limits.

iv) If M with filtration FM has completion $\widehat{M}$, then the canonical (filtered) morphism $\Psi_M : M \to \widehat{M}$ induces a graded isomorphism $G(\psi_M) : G(M) \to G(\widehat{M})$, where $\widehat{M}$ has the canonical filtration defined by FM.

Proof We leave it as an exercise (only iv. requires some work). □

A sequence $L \underset{f}{\longrightarrow} M \underset{g}{\longrightarrow} N$ is called **strict exact** in R-filt if it is an exact sequence of filtered R-modules such that both f and g are strict morphisms.

3.2.20 Theorem

In R-filt let $(*)$ $\quad L \underset{f}{\longrightarrow} M \underset{g}{\longrightarrow} N$ be such that $g \circ f = 0$. Consider the associated sequence in $G(R)$-gr :

$G(*)$: $$G(L) \underset{G(f)}{\longrightarrow} G(M) \underset{G(g)}{\longrightarrow} G(N)$$

Then we have the following

1. If $*$ is strict exact then $G(*)$ is exact.
2. If $G(*)$ is exact then g is strict
3. If $G(*)$ is exact, L is complete and FM is separated, then f is strict.
4. If $G(*)$ is exact and FM is discrete then f is strict.
5. If L is complete and FM is separated, or if FM is discrete, then $(*)$ is strict exact if and only if $G(*)$ is.

Proof

1. It is clear that $G(g) \circ G(f) = 0$. If we take $x \in F_nM$ such that $G(g)$ $(x \bmod F_{n-1}M) = 0$ then $g(x) \mathrm{mod} F_{n-1}N = 0$ or $g(x) \in F_{n-1}N$. Since g is assumed to be strict there is an $x' \in F_{n-1}M$ such that $g(x) = g(x')$, i.e. $x - x' = f(y)$ for some $y \in F_nL$. Consequently, $G(f)$ $(y \bmod F_{n-1}L) = x \bmod F_{n-1}M$ and therefore $\mathrm{Im}G(f) = \mathrm{Ker}G(g)$ and exactness of $G(*)$ follows.

2. Pick $y \in (F_nN \cap \mathrm{Im}g) - F_{n-1}N$. There is an $x \in M$ such that $g(x) = y$, say $x \in F_{n+s}M$ for some $s \geq 0$. If $s = 0$ there is nothing to prove. If $s > 0$, then $G(g)(x \bmod F_{n+s-1}M) = 0$. The exactness of $G(*)$ implies that $x \mathrm{mod} F_{n+s-1}M = G(f)(z \mathrm{mod} F_{n+s-1}L)$ for some $z \in F_{n+s}L$. It then follows that $x - f(z) \in F_{n+s-1}M$ and $y = g(x) = g(x - f(z)) = g(x')$ with $x' \in F_{n+s-1}M$. Repetition of this argumentation leads to an $m \in F_nM$ such that $y = g(m)$.

3. Look at $y \in F_nM \cap \mathrm{Im}f$. By the exactness of $G(*)$ we obtain :

   $G(g)(y \bmod F_{n-1}M) = 0$, thus $y \bmod F_{n-1}M == G(f)(x^n \mathrm{mod} F_{n-1}L)$ for some $x^n \in F_nL$. Hence $y - f(x^n) \in \mathrm{Im}f \cap F_{n-1}M$. By induction we obtain a sequence $x^n, x^{n-1}, \ldots, x^{n-s}$ with $x^{n-s} \in F_{n-s}L$, such that : $y - f(x^n) - \ldots - f(x^{n-s}) \in \mathrm{Im}f \cap F_{n-s-1}M$. By the completeness of L we may define an element $x = \sum_{s=0}^{\infty} x^{n-s}$ in F_nL. Then we arrive at : $y - f(x) = y - \lim\limits_{s \to \infty} f(x^n + \ldots + x^{n-s}) = 0$, the latter following from the separatedness of FM. Therefore we find a $y \in f(F_nL)$ and $f(F_nL) \subset F_nM \cap \mathrm{Im}f$ is obvious.

4. Along the lines of 3. but using $F_{n-s-1}M = 0$ for some (big) s.

5. Strict exactness of $(*)$ implies exactness of $G(*)$ because of 1. On the other hand if $G(*)$ is exact then we are either in the situation of 3. or 4., so in any case f is strict. By 2, also g is strict. If $y \neq 0$ in M is such that $g(y) = 0$, then $y \in F_nM - F_{n-1}M$ by separatedness of M. This leads to $G(g)(y \bmod F_{n-1}M) = 0$; now we go on as in the proof of 3. (resp. 4,) to find $y = f(x)$ (where $x = x^n + x^{-n-1} + \ldots + x^{n-s}$ or $x = \sum_{s=0}^{\infty} x^{n-s}$ depending on the case). Thus $(*)$ is exact. □

3.2.21 Corollary

Let $f : M \to N$ be a filtered morphism with FM and FN being separated.

1. The $G(f)$ is injective if and only if f is injective and strict.

2. If M is complete, then $G(f)$ is an isomorphism in $G(R)$-gr if and only if f is an isomorphism in R-filt.

3. Either if FM is complete or discrete, then $G(f)$ is surjective if and only f is surjective and strict.

The foregoing results apply in case we have a positively filtered algebra, e.g. the Weyl algebras, and look at a finitely filtered A-module with so-called **good filtration** i.e. there are $m_1, \ldots, m_d$ in M such that for all $n \in \mathbb{Z}$, $F_nM = \sum F_{n-d_i}Am_i$ for some $d_1, \ldots, d_n \in \mathbb{Z}$.

We will now see how filtered rings appear by dehomogenizations of $\mathbb{Z}$-graded rings, which will lead to the construction of $\mathbb{Z}$-graded rings or blow up rings, that will define the original filtered rings by dehomogenization.

In projective algebraic geometry, homogeneous coordinate rings appear together with some dehomogenization. If $V(I)$ is a projective variety determined by a homogeneous prime ideal of the polynomial ring $K[X_0, \ldots, X_n]$, where K is an algebraically closed field, and $R = K[X_0, \ldots, X_n]/I$ is the graded coordinate ring. Then $A = R/(1 - x_0)R$, where $x_0 = X_0 \mathrm{mod}\ I$, is isomorphic to the coordinate ring of the open affine subvariety complementary to the part at infinity (defined by the vanishing of x_0) in $V(I)$. In a similar way every determinantal ring is a dehomogenization of a Schubert cycle (being the graded coordinate ring of a Schubert variety). The dehomogenization principle is the basis for the study of determinantal rings, cf. W. Burns, U. Vetter ([3], LNM 1327, Springer Verlag 1988) for full detail on these geometric aspects we will not treat in these lecture notes.

Let $S = \oplus_{n\in\mathbb{Z}} S_n$ be a $\mathbb{Z}$-graded ring and T a homogeneous central element in S. The quotient ring $S/(T-1)S = R$ has a filtration induced by the gradation (filtration) of S, $F_nR = (S_n+(1-T))/(1-T)n \in \mathbb{Z}$. Similarly if M is a graded S-module then $F_n\overline{M} = M_n + (1-T)M/(1-T)M$, for $n \in \mathbb{Z}$, puts on $\overline{M} = M/(T-1)M$ the structure of a filtered R-module. We call R the **dehomogenization of** S **with respect to** T, $\overline{M}$ the **dehomogenization** of M, and this is an exhaustive filtration. The foregoing definitions remain unchanged if T is only a normalizing element i.e. $ST = TS$.

3.2.22 Lemma

Let T be a central regular homogeneous element of S of degree one then : $S(1-T)\cap S_n = 0$, for $n \in \mathbb{Z}$; if M is a T-torsionfree (i.e. $Tm = 0$ only if $m = 0$) graded S-module then $(1-T)M \cap M_n = 0$ for all $n \in \mathbb{Z}$.

Proof If $s_n \in S_n$ is in $S(1-T)$ then $s_n = s(1-T)$ for some $s \in S$. If $s = s_{n_1} + \ldots + s_{n_d}$ with $n_1 > \ldots > n_d$ is the homogeneous decomposition of s then : $s_n = -s_{n_1}T$ as the latter is nonzero and it is the homogeneous component of highest degree in $s(1-T)$ a contradiction. $\square$

3.2.23 Proposition

With notation as before :

1. $G(R) \cong S/TS$ as graded rings.

2. The localization of S at the central Ore set $\{1, T, \ldots, T^n, \ldots\}$ denoted by $S_{(T)}$ is isomorphic to $(S_{(T)})_0[T, T^{-1}]$. Moreover there is a commutative diagram of ring homomorphisms :

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "\pi"'] & & S_{(T)} \arrow[dl, "\psi"] \\ & R & \end{tikzcd}$$

   where π is the canonical map modulo $(1-T)S$ and $\Psi(T^js) = s + (1-T)S$, note that $S_{(T)}$ is $\mathbb{Z}$-graded by $(S_{(T)})_n = \{\sum_j T^j s_{n-j}, j \in \mathbb{Z}, s_{n-j} \in S_{n-j}, j \in \mathbb{Z}\}$, for $n \in \mathbb{Z}$.

3. ψ maps $(S_{(T)})_0$ isomorphically to R.

Proof

1. By definition : $G(R) = \oplus_{n\in\mathbb{Z}}(S_n + (1-T)S)/(S_{n-1} + (1-T)S)$, $S/TS = \oplus_{n\in\mathbb{Z}}(S_n + TS)/TS$.

   For each $n \in \mathbb{Z}$ we define $\varphi_n : G(R)_n \to (S_n + TS)/TS$, $s_n + S_{n-1} + (1-T)S \mapsto s_n + TS$. The φ_n are isomorphisms of additive groups, indeed for $s_{n-1} \in S_{n-1}$ we have that $s_{n-1} = Ts_{n-1} + (1-T)s_{n-1}$, hence φ_n is well defined. Moreover if $s_n \in TS$ then $s_n = Ts_{n-1}$ for some $s_{n-j} \in S_{n-1}$ and then $s_n + S_{n-1} + (1-T)S$ is the zero class in $G(R)_n$, hence φ_n is injective. That φ_n is surjective and an additive group morphism is obvious. The $\varphi_n, n \in \mathbb{Z}$, define a ring isomorphism $\oplus_{n\in\mathbb{Z}}G(R)_n \to S/TS$ which is by definition graded.

2. Since T is homogeneous of degree 1, $S_{(T)}$ is strongly graded and isomorphic to $(S_{(T)})_0[T, T^{-1}]$.

3. Since $S_{(T)}$ is $\mathbb{Z}$-graded and T is central homogeneous of degree one in $S_{(T)}$, the Lemma 3.2.23 implies that $(1-T)S_{(T)} \cap (S_{(T)})_0 = 0$ hence the restriction of Ψ to $(S_{(T)})_0$ is injective, since $\Psi|S$ is surjective an $r \in F_nR$ being image of an $s_n \in S_n$ thus of $s_nT^{-n} \in (S_{(T)})_0$. Thus Ψ maps $(S_{(T)})_0$ isomorphically to R. □

In formally the same way we prove the following

3.2.24 Proposition

Let M be a T-torsionfree graded S-module and $\overline{M} = M(1-T)M$, then :

1. $G(M) \cong M/TM$ as graded $G(R)$-modules.

2. The localization of M at the central Ore set $\{1, T, T^2, \ldots\}$ is denoted by $M_{(T)}$ and the natural homomorphism $\pi : M \to \overline{M}$ factors though $M_{(T)}$, yielding a commutative S-module morphisms

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "\pi"'] & & M_{(T)} \arrow[dl, "\psi"] \\ & \overline{M} & \end{tikzcd}$$

   where $M_{(T)}$ is a graded $S_{(T)}$-module and $\Psi(T^j m)$ is $m + (1-T)M$. Since $S_{(T)}$ is strongly graded $M_{(T)} \cong S_{(T)} \otimes_{(S_{(T)})_0} (M_{(T)})_0$.

3. The homomorphism Ψ maps $(M_{(T)})_0$ isomorphically to M. □

3.2.25 Definition

Let R be ring with a separated filtration FR. Let T be a symbol commuting with the elements of R and look at the ring $R[T, T^{-1}]$. Define $\widetilde{R}_n = F_nRT^n$ for $n \in \mathbb{Z}$, $\widetilde{R} = \oplus_{n\in\mathbb{Z}}\widetilde{R}_n$. Then $\widetilde{R}$ is a graded subring of $R[T, T^{-1}]$ with $\mathbb{Z}$-gradation defined by $R[T, T^{-1}]_n = RT^n$, for $n \in \mathbb{Z}$.

We call $\widetilde{R}$ the **Rees ring** (or **blow-up ring**) of FR. Let M be a filtered R-module with separated filtration FM. Then we may define the **Rees module** $\widetilde{M}$ as $\oplus_{n\in\mathbb{Z}}F_nM$ which we may view as $\widetilde{M} \subset M[T, T^{-1}] = R[T, T^{-1}] \otimes_R M$, $\widetilde{M} = \sum_{n\in\mathbb{Z}} F_nMT^n$, where F_nMT^n coincides with $T^n \otimes_R F_nM$. For $\widetilde{a} \in F_pRT^p, \widetilde{m} \in F_\delta MT^\delta$ we have $\widetilde{am} \in F_{p+\delta}MT^{p+\delta}$, the product $\widetilde{a}.\widetilde{m}$ in $\widetilde{M}$ therefore corresponds with the "tensorproduct" in $M[T, T^{-1}]$.

3.2.26 Remarks

1. Since $1 \in F_0R \subset F_1R$ we have $T \in \widetilde{R}_1$, hence T is central homogeneous of degree one in $\widetilde{R}$.

2. The element T is a regular element in $\widetilde{R}$. This follows easily from the fact that $\widetilde{R}$ is a subring of $R[T, T^{-1}]$.

3. $\widetilde{R}T$ is a graded ideal of $\widetilde{R}$ and since $1 - T$ is a central element also $\widetilde{R}(1 - T)$ is an ideal.

3.2.27 Proposition

With notation as above :

1. $\widetilde{R}/\widetilde{R}T \cong G(R)$ as graded rings.

2. $\widetilde{R}/\widetilde{R}(1 - T) \cong R$ as filtered rings, that is, R is the dehomogenization of $\widetilde{R}$ with respect to $T \in \widetilde{R}_1$.

Proof

1. Follows from Proposition 3.2.24.1. and 2. hereafter, we include a direct proof. Observe that $\widetilde{R}/\widetilde{R}T = \oplus_{n\in\mathbb{Z}} \widetilde{R}_n/(\widetilde{R}T)_n = \oplus_{n\in\mathbb{Z}} F_nRT^n/F_{n-1}RT^n = \oplus_{n\in\mathbb{Z}}(F_nR/F_{n-1}R)T^n$. We may thus define an additive bijection : $\pi : \widetilde{R}/\widetilde{R}T \to G(R)$, $\overline{a}_nT^n \mapsto \overline{a}_n$, where $\overline{a}_n \in G(R)_n$. It is clear that π is a morphism of graded algebras because for homogeneous elements we have :

$$\begin{aligned}&\pi((a_n \mathrm{mod} F_{n-1}RT^n)(b_m \mathrm{mod} F_{m-1}RT^m))\\ &\qquad = \pi(a_nb_m \mathrm{mod} F_{n+m-1}RT^{n+m}) = (a_n \mathrm{mod} F_{n-1}R)(b_m \mathrm{mod} F_{m-1}R)\end{aligned}$$

2. Look at the following diagram :

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{R}(1-T) \arrow[r, hook] \arrow[d, hook] & \widetilde{R} \arrow[r] \arrow[d, hook] & \widetilde{R}/\widetilde{R}(1-T) \arrow[r] & 0 \\
0 \arrow[r] & R[T,T^{-1}](T-1) \arrow[r, hook] & R[T,T^{-1}] \arrow[r, "\pi"] & R[T,T^{-1}]/R[T,T^{-1}](1-T) \arrow[r] \arrow[d, equal] & 0 \\
& & & R &
\end{tikzcd}$$

Here $\widetilde{R}$ is embedded via $\widetilde{R} \hookrightarrow R[T, T^{-1}]$ so we may restrict π to $\widetilde{R}$. The morphism $\pi|\widetilde{R}$ is surjective because for $a \in F_nR$ we have that $aT^n \in \pi^{-1}(a)$. The kernel of $\pi|\widetilde{R}$ contains $\widetilde{R}(T - 1)$, thus it follows that $\phi : \widetilde{R}/\widetilde{R}(1 - T) - R$, $a\mathrm{mod}(T - 1) \mapsto \pi(a)$, is well defined. In fact ϕ is a bijection. Injectivity of ϕ follows from $(1 - T)R[T, T^{-1}] \cap \widetilde{R} = (1 - T)\widetilde{R}$; indeed $a(T - 1) \in (T - 1)R[T, T^{-1}] \cap \widetilde{R}$, for $a = r_kT^k + \ldots + r_nT^n$, yields $r_{i-1} - r_i \in F_iR$ and $r_k \in F_kR$, then by induction $r_i \in F_iR$ and $a \in \widetilde{R}$ or $a(T - 1) \in (T - 1)\widetilde{R}$. Since ϕ is obviously surjective and a filtered algebra morphism we find that $R = \widetilde{R}/(1 - T)\widetilde{R}$ or R is the dehomogenization of $\widetilde{R}$. $\square$

3.2.28 Observation

The filtration FR is defined by the gradation of $\widetilde{R}$ via $R \cong \widetilde{R}/(1-T)\widetilde{R}$.

Proof Look at the gradation filtration on $\widetilde{R}$,

$$F_n^g \widetilde{R} = \oplus_{m \le n} \widetilde{R}_m = \oplus_{m \le n} F_m R T$$

Then we have : $F_n^g \widetilde{R} \bmod \widetilde{R}(1-T) = \bigcup_{m \le n} F_m R = F_n R$. □

This learns that the properties of a filtered rings R and its associated graded ring $G(R)$ are traceable via $\widetilde{R}$.

3.2.29 Proposition

Let M be a filtered R-module with filtration FM. With notation as before, we write $\widetilde{M}$ for the graded $\widetilde{R}$-module corresponding to FM.

1. $\widetilde{M}/T\widetilde{M} \cong G(M)$ as graded $G(R)$-modules.
2. $\widetilde{M}/(1-T)\widetilde{M} \cong M$ as filtered R-modules.

Proof Straightforward modification of the ring case above. □

The class of T-torsionfree graded $\widetilde{R}$-modules is a full subcategory of $\widetilde{R}$-gr, denoted by $\mathcal{F}_T$. The functor $\sim$: R-filt $\to \widetilde{R}$-gr is given by $M \mapsto \widetilde{M}$, it defines an equivalence of categories between R-filt and $\mathcal{F}_T$. The functor $G = \widetilde{R}/T\widetilde{R} \otimes_{\widetilde{R}} - : \mathcal{F}_R \to G(R)$-gr is exact on $\mathcal{F}_T$ and the functor D, $D = \widetilde{R}/(1-T)\widetilde{R} \otimes_{\widetilde{R}} - : \widetilde{R}$-gr$\to R$-filt is exact. These properties are easy to verify if one has a basic knowledge of category theory, we do not go into this here, but we just mention one related result.

3.2.30 Proposition

Let R be filtered with filtration FR.

1. If $f \in \mathrm{Hom}_{FR}(M,N)$ is a strict filtered morphism for $M, N \in R$-filt, then $\mathrm{Ker}\widetilde{f}$ and $\mathrm{Coker}\widetilde{f}$ are in $\mathcal{F}_T$, where $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ may be obtained from $\widetilde{f}(\widetilde{m}_n T^n) = f(\widetilde{m}_n)T^n$, $\widetilde{m}_n \in F_n M$.
2. If M, N, K are filtered R-modules and we have a sequence in R-filt :

   (∗) $$M \xrightarrow[f]{} N \xrightarrow[g]{} K$$

   Then let

   (∗∗) $$\widetilde{M} \xrightarrow[\widetilde{f}]{} \widetilde{N} \xrightarrow[\widetilde{g}]{} \widetilde{K}$$

   be the corresponding sequence in $\widetilde{R}$-gr. Then if (∗∗) is exact then (∗) is exact and f is strict. Conversely, if (∗) is exact and f is strict then (∗∗) is exact.

3.3 The Weyl Algebra and Some Examples

In general, if a filtered ring R is given by generators and relations, i.e. by an exact sequence :

$$0 \to \mathcal{R} \to F_K(X) \underset{\pi}{\longrightarrow} R \to 0$$

where X is a set of free variables and $F_K(X)$ is the free K-algebra generated by X and such that $\pi(X)$ generates R as a K-algebra, then the filtration of R is defined by the grading filtration of $F_K < X >$ and all maps in the sequence are strict filtered homomorphisms (of filtered $F_K < X >$-modules. Therefore we obtain an exact sequence of graded $F_K(X)^\sim$-modules :

$$0 \to \widetilde{R} \to F_K < X >^\sim \to \widetilde{R} \to 0$$

Now it is easy to see that $\widetilde{R}$ is a two-sided ideal and $\widetilde{R}$ is a ring so that $\widetilde{R}$ coincides with the Rees ring of FR. The Rees ring of $F_K < X >$ is $F_{K[T]} < X >= F_K < X > [T]$, this follows from some observations regarding gradation filtrations. Consider a $\mathbb{Z}$-graded ring R and the polynomial ring $R[T]$ over R with variable T. For every $k \in \mathbb{Z}$ we define : $R[T]_k^+ = \oplus_{n\geq 0} R_{k+n}T^n$, $R[t]_k^- = \oplus_{n\geq 0} R_{k-n}T^n$.

3.3.1 Lemma

i) $R[T] = \oplus_{k\in\mathbb{Z}} R[T]_k^+ = \oplus_{k\in\mathbb{Z}} R[T]_k^-$.

ii) The $R[T]_k^+$, resp. $R[T]_k^-$, define a $\mathbb{Z}$-gradation on $R[T]$. Then $R[T]_0^+ = \oplus_{n\geq 0} R_n T^n \cong R^+ = \oplus_{n\geq 0} R_n$ and $R[T]_0^- = \oplus_{n\geq 0} R_{-n}T^n \cong R^- = \oplus_{n\geq 0} R_{-n}$.

iii) If R is strongly $\mathbb{Z}$-graded then $R[T]$ is strongly $\mathbb{Z}$-graded both with respect to the $+$ or $-$ gradation defined above.

Proof The proof is obvious. □

For any $\mathbb{Z}$-graded ring the gradations $+$ or $-$ defined on $R[T]$ are called the **sign gradations**. On the other hand for every $\mathbb{Z}$-graded ring R there are two natural filtrations on R defined by the gradation as follows :

$$F_n^{(1)}R = \oplus_{k\leq n} R_k, n \in \mathbb{Z}, \text{ or, } F_n^{(2)}R = \oplus_{k\geq n} R_n, n \in \mathbb{Z}$$

We call these filtrations the grading filtrations $F^{(1)}R$, resp. $F^{(2)}R$.

3.3.2 Proposition

With notation as above write $\widetilde{R}^{(1)}$, resp. $\widetilde{R}^{(2)}$, for the Rees ring with respect to $F^{(1)}R$ resp. $F^{(2)}R$. Then we have isomorphisms of graded rings $\widetilde{R}^{(1)} \cong R[T]$ where $R[T]$ has the $-$gradation, $\widetilde{R}^{(2)} \cong R[T]$ where $R[T]$ has the $+$-gradation.

Proof For each $k \in \mathbb{Z}$ we have isomorphisms of additive groups :

$$\oplus_{n\geq 0} R_{k-n}T^n = R[T]_k^- \to \widetilde{R}_k^{(1)} = F_k^1 R = \oplus_{i\leq k} R_i$$

this leads in the obvious way to a graded ring isomorphism $R[T] \cong \widetilde{R}^{(1)}$. □

3.3.3 The Bernstein Filtration of the n-th Weyl Algebra

The Rees algebra of $\mathbb{A}_n(K)$ with respect to the Bernstein filtration is isomorphic to the graded subring : $K < x_1T, \dots, x_nT, y_1T, \dots, y_nT, T >$ of the polynomial ring $\mathbb{A}_n(K)[T]$ over $\mathbb{A}_n(K)$. For $n = 1$ we obtain the Rees ring $\widetilde{\mathbb{A}_1(K)}^B \cong K < X, Y > [T]/(YX - XY - T^2)$. Of course the associated graded $G_B(\mathbb{A}, (K)) \cong K[X, Y]$.

3.3.4 The Σ-filtration

In Proposition 2.1.6. we have seen that there is a cannonical epimorphism $U_K(\mathcal{H}_n) \twoheadrightarrow \mathbb{A}_n(K)$, where $\mathcal{H}_n$ is the $2n{+}1$-dimensional Heisenberg algebra over K with basis $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ and brackets $[x_i, y_i] = z$ for $i = 1, \dots, n$ and all other basis brackets zero. Putting deg $x_i = 0, \deg\ y_i = 1$ for $i = 1, \dots, n$, then $x_i y_i - y_i x_i = z$ is homogeneous, hence $U_K(\mathcal{H}_n)$ is a $\mathbb{Z}$-graded algebra having z as a central homogeneous regular element of degree 1. We have $\mathbb{A}_n(K) = U_K(\mathcal{H}_n)/(z - 1)$ and thus, by Proposition 3.2.27.2. $U_K(\mathcal{H}_n)$ is the Rees ring of $\mathbb{A}_n(K)$ with the Σ-filtration. Of course $G_\Sigma(\mathbb{A}_n(K))$ is again $K[X, Y]$, this time with $\deg X = 0, \deg Y = 1$.

For the Bernstein filtration : $[y_1T, x_1T] = T^2$, hence $\widetilde{\mathbb{A}_1(K)}^B i = K < x, T, y_1T, T^2 > [T] \cong U(\mathcal{H}_\xi)[t]$ with $t^2 = Z$ (putting $x_1T = X, y_1T = Y, T^2 = Z$), i.e. a quadratic extension of $U(\mathcal{H}_\xi)$.

In particular we have that $\widetilde{\mathcal{R}}^B = (YX - XY - T^2)$ and $\widetilde{\mathbb{A}_1(K)}^B = K < X, Y > [T]/(YX - XY - T^2)$. On the other hand $\widetilde{\mathcal{R}}^\Sigma = (YX - XY - T)$ and $\widetilde{\mathbb{A}, (K)}^\Sigma = K < X, Y > [T]/(YX - XY - T)$. In general $\widetilde{\mathcal{R}}$ is the Rees (ideal) module associated to $F\mathcal{R}$ but it need not be generated by the homogenizations of generators for $\mathcal{R}$ (as a left ideal or as an ideal). For the Weyl algebras this does hold. Another case where this holds is for the standard filtration of $U_K(g)$ where g is any finite dimensional Lie algebra over K. We have $U_K(g) = F_K < g > /(x_i.x_j - x_j.x_i - \sum_k c_{ij}^k x_k)$ where $\{x_1, \dots, x_n\}$ is a K-basis for g and the c_{ij}^k are the structure constants : $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$. We consider the homogeneous Lie algebra $\overline{g}$ with basis $X_i, i = 1, \dots, n$ and $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k T X_k$ considered over $K[T]$.

We have a $K[T]$-enveloping algebra $U_{K[T]}(\overline{g})$ which is obviously a $\mathbb{Z}$-graded ring where we have $\deg X_i = \deg T = 1$ for $i =, \dots, n$ and T is a central homogeneous regular element in $U_{K[T]}(\overline{g})$.

3.3.5 Example

The Rees ring of $U_K(g)$ is $U_{K[T]}(\overline{g})$.

Proof Sending X_i to x_i and T to 1 defines a ring epimorphism

$$U_{K[T]}(\overline{g}) \twoheadrightarrow U_K(g)$$

because by the PBW-theorem the monomials in the x_i, resp. the monomials in X_i form a K-basis, resp. a $K[T]$-basis, for $U_K(g)$, resp. $U_{K[T]}(\overline{g})$, while the $c_{ij}^k T$ maps to c_{ij}^k. By Proposition 2.1.6., $U_K(g)$ is the dehomogenization of $U_{K[T]}(\overline{g})$, hence the latter is the Rees ring of $U_K(g)$ with the standard filtration. Since $U_{K[T]}(\overline{g})$ is equal to $K[T] < X, Y > /(X_iX_j - X_jY_i - c_{ij}^k T X_k)$ it is obtained from $U_K(g)$ by homogenization of the relations.

3.3.6 Generalized Rees Rings

Let R be a ring and I an ideal of R; consider an overring S of R such that there is an R-subbimodule J of S such that $IJ = JI = R$. Then I is said to be **invertible in** S. We write $J = I^{-1}$; indeed J is unique because if J_1 is another R-subbimodule of S such that $J_1 I = I J_1 = R$ then $J_1 IJ = RJ = J$ and $J_1 IJ = J_1 R = J_1$, so $J = J_1$. Now we consider the I-adic filtration $F_n R = I^{-n}$ for $n < 0$, $F_n R = R$ for $n \geq 0$. Consider $R(I) = \oplus_{n \geq 0} I^n T^n \subset R[T]$. Then $R(I)$ is as a graded ring isomorphic to the negative part $\widetilde{R}^- = \oplus_{n \leq 0} I^{-n}$ of the Rees ring $\widetilde{R}$ for the I-adic filtration. We may define a filtered overring of R defined by $S(I) = \cup_{n \in \mathbb{Z}} I^n \subset S$. Then there is another associated graded ring $\check{R}(I) = \oplus_{n \in \mathbb{Z}} I^n T^n \subset S[T_1 T^{-1}]$, which is isomorphic to the Rees ring of $S(I)$. We call $\check{R}(I)$ the **generalized Rees ring of** R **with respect to** I. Note that $\check{R}(I)_0 = R$ and $T^{-1} \in \check{R}(I)_1$ (the gradation of $\check{R}(I)$ is defined by $\check{R}(I)_n = I^{-n} T^{-n}$), T^{-1} is a regular central homogeneous element of degree 1 in $\check{R}(I)$. Thus we have a dehomogenization $A = \check{R}(I)/(1 - T^{-1})\widetilde{R}(I)$. Let us write $Y = T^{-1}$. Look at the localization $\check{R}(I)_{(Y)}$ at the central Ore set $\{I, Y, Y^2, \ldots, Y^n, \ldots\}$.

Then we have the following properties :

a. $A \cong S(I) \cong (\widetilde{R}(I)_{(Y)})_0$

b. $G(A) \cong \check{R}(I)/Y R(I)$ and $G(A)_0 \cong R(I)$

c. $G(A)^- = \oplus_{n \leq 0} G(A)_n \cong \oplus_{n \geq 0} I^n / I^{n+1} = G_I(R)$, where $G_I(R)$ stands for the associated graded ring of R with respect to the I-adic filtration on R.

Generalized Rees rings have several applications, we refer to [12] for some of these.

3.4 Right or wrong

1. A finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded algebra has necessarily even dimension.
2. Every graded algera is a homomorphic image of a graded free algebra.
3. Let $\mathcal{S}_n$ be the permutation group on n-elements. On the polynomial ring $K[X]$ there is an $\mathcal{S}_3$-gradation such that $K[X]_\sigma \neq 0$ for all $\sigma \in \mathcal{S}_3$.
4. If N is a normal subgroup of G, then $R_N = \oplus_{n \in N} R_n$ is an ideal of the G-graded ring R.
5. Is $\oplus_{g \neq e} R_g$ an ideal of the G-graded ring R ?
6. Suppose $U_{\mathbb{C}}(g)$ is $\mathbb{Z}$-graded such that for all $x \in g \subset U_{\mathbb{C}}(g)$ x has degree 1, then g is abelian.
7. The group algebra $K(G)$ is G-graded by giving u_g degree g.
8. If $a \in R$, R a G-graded ring, is homogeneous and invertible then a^{-1} is also homogeneous.
9. The field $\mathbb{C}(X)$ has a $\mathbb{Z}$-gradation such that $\mathbb{C}(X)_k = \mathbb{C}X^k$ for $k \in \mathbb{Z}$.
10. Every graded algebra has a basis of homogeneous elements.

11. A basis of a graded algebra cannot contain exactly one non-homogeneous element.
12. The direct sum of $\mathbb{Z}$-graded algebras A, B can be $\mathbb{Z}$-graded by putting : $(A \oplus B)_i = A_i \oplus B_i, i \in \mathbb{Z}$.
13. An ideal generated by non-homogeneous elements is never a graded ideal.
14. If a finite dimensional K-algebra is G-graded and a field then G is a finite group.
15. The matrix algebra cannot be nontrivially graded by $\mathbb{Z}_2$.
16. A finite dimensional $\mathbb{Z}$-graded algebra contains nilpotent elements.
17. A path-algebra can be $\mathbb{Z}$-graded by giving every arrow an arbitrary degree.
18. The algebra $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$ cannot be graded nontrivially by $\mathbb{Z}$.
19. The number of different $\mathbb{Z}_2$-gradations on a finite dimensional algebra is finite.
20. Any G-graded K-algebra A with $A_g \neq 0$ for all $g \in G$ may be mapped to $K[G]$ surjectively.
21. The exterior algebra has a natural $\mathbb{Z}_z$-gradation.
22. Let R be a nontrivially $\mathbb{Z}$-graded algebra; if $e^2 = e$ in R, then $e \in R_0$.
23. The skew polynomial ring $K[X, \varphi]$ cannot be $\mathbb{Z}$-graded positively such that some elements of K have a strict positive degre.
24. Every invertible element in a G-graded ring has degree e.
25. If a subset of one element in an algebra A is closed in the filtration topology, then every element of A is closed.
26. Every automorphism of the algebra $K[X]$ is continuous in the standard filtration topology.
27. The filtration topology of the p-adic numbers is connected.
28. If the filtration topology of an algebra is compact then the algebra is finite.
29. The real numbers cannot be viewed as the completion of $\mathbb{Q}$ according to a filtration on $\mathbb{Q}$.
30. Every element in the (2)-adic completion of $\mathbb{Z}$ is uniquely expressed as a power series of powers of 2 with coefficients 0 and 1.
31. The associated graded ring of the X-adic filtration on $K[X]$ is isomorphic to $K[X]$.
32. The set of zero-divisors of a ring is an open set in some filtration topology.
33. The set of zero divisors in a ring is a closed set in some filtration topology.
34. Look at the ideal N of nilpotent elements in a finitely generated commutative algebra A. The N-adic filtration defines the discrete topology.

35. A matrix algebra does not have a nontrivial $\mathbb{Z}$-gradation.

36. We have $G(U_K(g)) = G(U_K(h))$ if and only if $g = h$, where $G(-)$ is calculated with respect to the standard filtration on $U_K(g)$, resp. $U_K(h)$.

37. $U_K(g)$ does not have zero-divisors.

38. The only invertible elements of $U_K(g)$ are elements of K.

39. If the sum of homogeneous elements is homogeneous then they have the same degree.

40. If the principal symbol map $\sigma : R \to G(R)$ is an algebra isomorphism then $F_0R = R$ (FR is separated).

41. If FR is a filtration on R then $F'R$ defined by $F'_{2n}R = F'_{2n+1}R = F_nR$ is again a filtration of R.

42. The Bernstein filtration and the operator filtration define the same topology on $\mathbb{A}_1(\mathbb{C})$.

43. The identification $\mathbb{C}^n \to \mathbb{C}[X]/(X^n - 1)$ is a continuous map, if we put on $\mathbb{C}^n$ the complex topology and on $\mathbb{C}[X]/(X^n - 1)$ the generator filtration topology.

44. If the topology of the I-adic filtration on R is Hausdorff then I does not contain an idempotent (I is not trivial).

45. The I-adic and the I^2-adic filtration define the same topology on R.

46. The Rees ring of a finite dimensional algebra is also a finite dimensional algebra.

47. Let R be a finitely generated filtered $\mathbb{C}$-algebra with Rees ring $\widetilde{R}$. The map fixing a set of generators of R but maps T to $\lambda T (\lambda \neq 0$ in $\mathbb{C})$ is an isomorphism of $\widetilde{R}$.

48. Can $\widetilde{R}$ be filtered in a natural way such that $\widetilde{\widetilde{R}} = \widetilde{R}$?

49. If the Rees ring of R is a polynomial algebra is then R also a polynomial ring ?

50. If a ring R can be embedded in its Rees ring, then R has to be a polynomial ring.

Chapter 4

Finiteness Conditions

4.1 Noetherian Conditions

Let R be a ring and M a left R-module. We say that M is **(left) Noetherian** if the submodules of M satisfy the ascending chain condition (a.c.c.), that is : a chain $M_0 \subset M_1 \subset M_2 \subset \ldots \subset M$, of left R-submodules becomes stationary, i.e. there is an $n_0 \in \mathbb{N}$ such that $M_{n_0} = M_{n_0+1} = \ldots$. We say that M **is (left) Artinian** if submodules of M satisfy the descending chain condition (d.c.c.) : $M \supset M_0 \supset \ldots \supset M_n \supset \ldots$ is stationary. The symmetric properties for right modules are defined in the obvious way. A ring R is left Noethertian, resp. left Artinian if the left R-module R is left Noetherian resp. left Artinian. Similar for right Noetherian and right Artinian. A ring is said to be Noetherian, resp. Artinian if it is simultaneously left and right Noetherian, resp. left and right Artinian.

4.1.1 Proposition

Look at an exact sequence in R-mod,

$$0 \longrightarrow N \longrightarrow M \underset{\pi}{\longrightarrow} M/N \longrightarrow 0$$

If M is left Noetherian then N and M/N are left Noetherian and conversely. Also, M is left Artinian if and only if N and M/N are left Artinian.

Proof Let us prove the statement about the Noetherian property, the proof of the Artinian case is similar.

First assume that M is left Noetherian. Then N is left Noetherian because an ascending chain of submodules of N is an ascending chain (*a.c.*) of left submodules of M. Look at an *a.c.* $L_0 \subset L_1 \subset \ldots \subset L_i \subset \ldots \subset M/N$ of left submodules of M/N and put $\pi^{-1}(L_i) = L_i + N, i = 0, 1, \ldots$. Then : $L_0 + N \subset \ldots \subset L_i + N \subset \ldots \subset M$ is an *a.c.* of left submodules of M, hence there is a $d \in \mathbb{N}$ such that $L_d + N = L_{d+1} + N = \ldots$. Since π is surjective we have $\pi(L_d + N) = \pi(L_{d+1} + N) = \ldots$ in M/N, hence $L_0 \subset L_1 \subset \ldots \subset L_i \subset \ldots$ terminates in M/N. Conversely assume N and M/N are left Noetherian and consider an *a.c.* $M_0 \subset M_1 \subset \ldots \subset M_i \subset \ldots \subset M$ of left submodules of M. Then $\pi(M_0) \subset \pi(M_1) \subset \ldots \subset \pi(M_i) \subset \ldots$ is an *a.c.* in M/N and thus there exists a $p \in \mathbb{N}$ such that : $\pi(M_p) = \pi(M_{p+1}) = \ldots$. Now for all q, $(M_q + N)/N = M_q/M_q \cap N$, thus we have : $M_p/M_p \cap N = M_{p+1}/M_{p+1} \cap N = \ldots$. Now we have an *a.c.* $M_0 \cap N \subset \ldots \subset M_i \cap N \subset \ldots \subset N$ of left submodules of $\mathbb{N}$, thus there exists an $r \in \mathbb{N}$ such that : $M_r \cap N = M_{r+1} \cap N = \ldots$. Taking q larger than p and r yields $M_q \cap N = M_{q+1} \cap N = \ldots$, as well as $M_q/M_q \cap N = M_{q+1}/M_{q+1} \cap N = \ldots$, hence $M_q = M_{q+1} = \ldots$ follows for all q larger than p and r, or the original *a.c.* becomes stationary.□

4.1.2 Corollary

1. The direct sum of a finite number of left Noetherian, resp. left Artinian, modules is itself left Noetherian, resp. left Artinian.

2. A finitely generated module over a left Noetherian ring is a left Noetherian module.

3. If I is an ideal of a left Noetherian resp. left Artinian ring R, then R/I is left Noetherian, resp. left Artinian.

Proof

1. Look at $M_1 \oplus \ldots \oplus M_d$, a direct sum of left Noetherian (resp. Artinian) modules. Then we have :$0 \to M_1 \oplus \ldots \oplus M_{d-1} \to M_1 \oplus \ldots \oplus M_d \to M_d \to 0$ is exact in R-mod, so we can argue by induction on d and conclude $M_1 \oplus \ldots \oplus M_d$ is left Noetherian (resp. Artinian).

2. If N is a finitely generated left R-submodulle over a left Noetherian ring R then there is a surjective R-linear map $R \oplus \ldots \oplus R \to N$, where there are finitely many terms R in the direct sum. In view of 1. we have that $R \oplus \ldots \oplus R$ is left Noetherian and Proposition 4.1.1. then yields that N is left Nooetherian (note that one may replace Noetherian by Artinian in 2. and the statement remains true).

3. Immediate from Proposition 4.1.1.. □

4.1.3 Proposition

Let M be a left R-module. The following statements are equivalent :

1. M is left Noetherian.

2. Every submodule N of M is finitely generated.

3. Every family of submodules of M has a maximal element.

Proof

- 1. $\Rightarrow$ 2. Suppose $N \subset M$ is not finitely generated, say with set of generators $\{x_1, x_2, \ldots\}$ an infinite set. Then $Rx_1 \subset Rx_1 + Rx_2 \subset \ldots \subset Rx_1 + \ldots + Rx_i \subset \ldots$ is an *a.c.*, hence it must become stationary, or $x_{d+k} \in Rx_1 + \ldots + Rx_d$ for every $k \in \mathbb{N}$; thus N is finitely generated.

- 2. $\Rightarrow$ 3. Let $\mathcal{F}$ be a nonempty family of submodules of M. Pick $M_1 \in \mathcal{F}$. If M_1 is not maximal then we can choose $M_2 \in \mathcal{F}$ and $M_2 \supset M_1$. If M_2 is not maximal, repeat the procedure. So we either find a maximal element or we obtain a chain $M_1 \subset M_2 \subset M_3 \subset \ldots \subset M_i \subset \ldots$. Look at $N = \cup M_i$. By the assumption N is finitely generated, say by $\{x_1, \ldots, x_d\}$ and thus there is an $n_0 \in \mathbb{N}$ such that $\{x_1, \ldots, x_d\} \subset M_{n_0}$. Thus $\cup_i M_i \subset M_{n_0}$ or $M_{n_0+k} = M_{n_0}$ for every $k \in \mathbb{N}$ and M_{n_0} is maximal.
- 3. $\Rightarrow$ 1. Consider an *a.c.* $M_0 \subset M_1 \subset \ldots \subset M$ and look at the family $\{M_i, i \in \mathbb{N}\}$. By 3. there is a maximal element M_{n_0} in this family and thus the *a.c.* terminates at M_{n_0}. □

4.1.4 Observation

In a similar way one can prove the equivalence of the following statements.

1. M is left Artinian.
2. Every family of submodules of M has a minimal element. □

4.1.5 Easy Examples

1. If $R = K$ is a field then it is obviously (left and right) Noetherian and Artinian.
2. A simple module, i.e. a module without nontrivial submodules, is automatically left Noetherian and left Artinian. A semisimple module is by definition a finite direct sum of simple modules, therefore any semisimple module is left Noetherian and left Artinian.
3. Every algebra A of finite dimension over a field K is Noetherian and Artinian. Indeed since proper left (or right) ideals have smaller dimension over K both the *a.c.c.* and *d.c.c.* are easily verified.
4. The ring of integers $\mathbb{Z}$ is Noetherian but not Artinian. Ideals of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ and $n\mathbb{Z} \subset m\mathbb{Z}$ if and only if m divides n. An ascending chain therefore corresponds to the set of divisors of an $m \in \mathbb{Z}$ corresponding to the beginning $m\mathbb{Z}$ of the chain; this set is finite so the *a.c.c.* holds. However a *d.c.* corresponds to taking multiples and this does not terminate.

4.1.6 Theorem (Hilbert)

If R is a left Noetherian ring then the ring of polynomoials $A = R[X_1, \ldots, X_n]$ is also a left Noetherian ring.

Proof It suffices to give the proof for $n = 1$ (by repetition).

For an ideal I of $R[X]$ we define :

$$I_k = \{a_k,\ \text{there is an } a_k X^k + \ldots + a_0 \in I\} \cup \{0\}$$

Clearly, I_k is a left ideal of R hence it is finitely generated by $\{a_{1k}, \ldots, a_{n_k k}\}$ by the left Noetherian property of R. For every generator a_{ij} there exists a polynomial $f_{ij}(X) = a_{ij}X^j + \ldots$

in I. Now $I_0 \subset I_1 \subset \ldots \subset I_i \subset \ldots$ is an *a.c.* in R (because an $a_iX^i + \ldots + a_0 \in I$ yields $a_iX^{i+1} + \ldots + a_0X \in I$ as I is an ideal of $R[X]$), hence the left Noetherian property of R yields $I_r = I_{r+1}$ for $r \geq m$, some $m \in \mathbb{N}$. If the $f_{ij}(X)$ do not generate I then we can choose $g = g_kX^k + \ldots + g_0 \in I - \Sigma R[X]f_{ij}(X)$ of lowest degree as such. We may write :

$$g_k = \sum_{j \leq \min(k,m)} c_{ij}a_{ij}, \text{ with } c_{ij} \in R$$

Hence

$$g(X) - \sum_{j \leq \min(k,m)} c_{ij}X^{k-j}f_{ij}(X) \in I$$

but it has degree strictly smaller than deg$g$, a contradiction. □

4.1.7 Remarks

1. The polynomial ring in an infinite number of variables is not Noetherian, in fact there will be a chain of left ideals $(X_1) \subset (X_1, X_2) \subset \ldots \subset (X_1, \ldots, X_i) \subset \ldots$ which cannot become stationary.

2. $K[X]$ where K is a field is not Artinian, the *d.c.* $(X) \supset (X^2) \supset \ldots \supset (X^i) \supset \ldots$ is not stationary.

3. A finitely generated left ideal of a ring R is not necessarily left Noetherian, e.g. take $L = R$.

4. From 4.1.3. it is clear that for a left Noetherian ring R a left module M is left Noetherian if and only if it is finitely generated.

Let I be an ideal of a ring R, then I is said to have the **finite intersection property** if for any finitely generated left R-module M we have :

$$\bigcap_{n \geq n} I^n M = \{m \in M, (1-a)m = 0, \text{ for some } a \in I\}$$

We say that I has the **Artin-Rees property** if for any finitely generated left R-module M, any submodule $N$of M and any $n \in \mathbb{N}$, there exists an integer $h(n) \geq 0$ such that $I^{h(n)}M \cap N \subset I^nN$; in other words I has the A.R. property if the I-adic topology of N coincides with the topology induced on N by the I-adic topology of M.

It is easy to see that if I has the $A.R.$-property, then it has the finite intersection property, indeed if $x \in \cap_{n \geq 1} I^n M$, then consider the submodule Rx of M and apply $A.R$ to it : $I^{h(n)}M \cap Rx \subset I^n x$ yields $Rx = I^n x$ or $x = ax$ for some $a \in I^n$, $(1-a)x = 0$. An $a \in R$ is normalizing if $aR = Ra$; the set of normalizing elements of N is denoted by $N(R)$. Let I be an ideal of R; a subset $\{a_1, \ldots, a_n\}$ is a normalizing (resp. centralizing) set of generators for I if

a. The ideal $(a_1, \ldots, a_n) = I$.

b. For $i = 1, \ldots, n, a_i \in N(R)$ (resp. $Z(R)$, the centre of R).

c. $a_i + (a_1, \ldots, a_{i-1}) \in N(R)/(a_1, \ldots, a_{i-1})$ (resp. $a_i + (a_1, \ldots, a_{i-1}) \in Z(R)/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, n$.

4.1.8 Lemma

Let R be left Noetherian, X a normalizing element of R and let M be a finitely generated X-torsion free R-module, then $\cap_{n\geq 1}X^nM = \{m \in M, (1-a)m = 0 \text{ for some } a \in RX\}$.

Proof Put S equal to $\{m \in M, (1-a), m = 0 \text{ some } a \in RX\}$. Obviously $S \subset \cap_{n\geq 1}X^nM$ (from $z \in S$ it follows $z = az$ for some $a \in RX$). Conversely take $y \in \cap_{n\geq 1}X^nM$, then $y = Xm_1 = X^2m_2 = \ldots$, where $X^km_k \in X^kM$. Since $M$is X-torsionfree $m_k = Xm_{k+1}, k = 1, 2, \ldots$, and consequently $Rm_1 \subset Rm_2 \subset \ldots$. Since R is left Noetherian and M is finitely generated we obtain : $Rm_n = Rm_{n+1}$ for all $n \geq n_0$, thus $m_{n+1} = rm_n = rXm_{n+1}$, or $X^{n+1}m_{n+1} = X^{n+1}rXm_{n+1} = Xr'X^{n+1}m_{n+1}$ for some $r, r' \in R$ since X is normalizing. Therefore : $0 = (1 - Xr')X^{n+1}m_{n+1} = (1 - Xr')y$, or $y \in S$. This shows $\cap_{n\geq 1}X^nM \subset S$ and thus equality holds. □

4.1.9 Lemma

Let R and X be as in foregoing lemma. For any finitely generated left R-module M then either $\cap_{n\geq 1}X^nM = 0$ or $\cap_{n\geq 1}X^nM = N$ is an X-torsionfree submodule of M.

Proof
It is clear that $\cap_{n\geq 1}X^nM$ is a submodule of M. Suppose that $\cap_{n\geq 1}X^nM \neq 0$. The X-torsion part $t(M) = \{m \in M, X^wm = 0 \text{ for some } w \geq 1\}$ is a finitely generated submodule of M such that $X^kt(M) = 0$ for some $k \in \mathbb{N}$ large enough. It follows that X^kM has no X-torsion elements, hence $\cap_{n\geq 1}X^nM$ is X-torsion free. □

4.1.10 Proposition

Let R and X be as above. The ideal RX of R has the finite intersection property.

Proof Consider a finitely generated R-module M and the exact sequence on R-mod : $0 \to t(M) \to M \to M/t(M) \to \circ$. Since $M/t(M)$ is X-torsionfree, foregoing lemmas entail :

$$\begin{aligned} \cap_{n\geq 1}X^n(M/t(M)) &\cong (\cap_{n\geq 1}(X^nM + t(M)/t(M) \\ &\cong (\cap_{n\geq 1}(X^nM + t(M))/t(M) \\ &\quad \cap_{n\geq 1}X^n(M/t(M)) \\ &\cong \{\overline{m} \in M)t(M), (1-a)\overline{m} = 0, \text{ some } a \in XR\} \end{aligned}$$

Thus, if $y \in \cap_{n\geq 1}X^nM$ there is an $a' \subset XR$ such that $(1-a')y = 0$, as desired.

□

4.1.11 Proposition

Let R be a left Noetherian ring.

1. If I is an ideal of R generated by a centralizing system then I has the Artin Rees (A-R) property.

2. If P is an invertible ideal of R, then P has the A-R-property.

3. If X is normalizing in R and also a regular element, then RX has the A-R-property.

Proof

1. Cf. [18] Proposition 5.1. (Dimension Theory)

2. Cf. [18], Lemma 6.5.2.

3. RX is an invertible ideal (i.e. $S = \{I, X, X^2, \ldots\}$ is an Ore set in R so we can look at $S^{-1}R$), then apply 2. □

We include the following without proof.

4.1.12 Proposition

Let I be an ideal of R generated by a centralizing system. If R/I is left Noetherian then the following statements are equivalent :

1. I has the finite intersection property

2. I satisfies the Artin Rees property

Now consider a $\mathbb{Z}$-graded ring A having a regular central homogeneous element of degree one.

4.1.13 Lemma

1. Let $\overline{M}$ be any T-torsionfree graded A-module and M the dehomogenization of $\overline{M}$ with filtration FM induced by the gradation of $\overline{M}$. Then $\cap_{n\geq 1} T^n\overline{M} = 0$ if and only if $\cap_{n\in\mathbb{Z}} F_n M = 0$, in other words : the AT-adic filtration on $\overline{M}$ is separated if and only if FM is separated.

2. $T \in J^g(A)$ if and only if $F_{-1}R \subset J(F_0R)$, where $R = A/(1-T)A$ and $J^g(A)$ is the graded Jacobson radical of A, $J(F_0R)$ being the Jacobson radical of F_0R.

Proof

1. Straightforward verification.

2. Suppose $T \in J^g(A)$, then $TA_{-1} \subset J^g(A) \cap A$, hence $1 - Ta_{-1}$ is invertible in A_0 for every $a_{-1} \in A_{-1}$. Then from $a_{-1} = Ta_{-1} + (1-T)a_{-1}$ it follows that : $(A_{-1} + (1-T)A)/(1-T)A = F_{-1}R \subset J(F_0R)$. The converse implication follows from Lemma 4.1.8. □

4.1.14 Corollary

Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM. Let T be the central homogeneous regular element of $A = \widetilde{R}$ of degree one.

1. We have $\cap_{n\geq 1} T^n \overline{M} = 0$ if and only if $\cap_{n\in\mathbb{Z}} F_n M = 0$.
2. We have $T \in J^g(\widetilde{R})$ if and only if $F_{-1}R \subset J(F_0R)$.

We say that FR is **faithful** if $F_{-1}R \subset J(F_0R)$.

4.1.15 Theorem

Suppose FR is faithful and $\widetilde{R}$ is left Noetherian. For a filtered R-module M such that $\widetilde{M}$ is finitely generated in $\widetilde{R}$-gr we have that FM is separated.

Proof Since $T \in J^g(\overline{R})$ and $\cap_{n\geq 1} T^n \widetilde{M}$ is a graded submodule of $\widetilde{M}$, hence finitely generated because $\overline{R}$ is left Noethertian. Corollary 4.1.11 (1) finishes the proof. □

4.2 Application to Modules over the Weyl Algebra

First a negative result.

4.2.1 Proposition

$\mathbb{A}_n(K)$ is not (left) Artinian.

Proof We give the proof for $n = 1$, the general case follows in the same way. Look at $x\mathbb{A}_1(K)$, i.e. those elements of $\mathbb{A}_1(K)$ written in the unique form with an x first. Observe that the inclusion $x^2\mathbb{A}_1(K) \subset x\mathbb{A}_1(K)$ is strict because $x = x^2z$ with $z \in \mathbb{A}_1(K)$ in the domain $\mathbb{A}_1(K)$ leads to $1 = xz$ and we know that x is not not invertible in $\mathbb{A}_1(K)$. We thus obtain a descending chain $\mathbb{A}_1(K) \supsetneq x\mathbb{A}_1(K) \supsetneq x^2\mathbb{A}_1(K) \supsetneq \ldots$ which does not become stationary. □

Recall that for a filtered submodule N of a filtered R-module M we have a quotient filtration $F(M/N)$ defined by $F_n(M/N) = (N + F_nM)/N = F_nM/(N \cap F_nM) = F_nM/F_nN$.

4.2.2 Lemma

Let $N \subset M$ in R-filt then for the quotient filtration $F(M/N)$ on M/N we have $G(M/N) \cong G(M)/G(N)$, where the latter is an isomorphism of $G(R)$ $\mathbb{Z}$-graded modules.

Proof We have that $G(N)$ is a graded submodule of $G(M)$ since $N \hookrightarrow M$ is a strict filtered morphism. We now calculate :

$$\begin{aligned} G_d(M/N) &= F_d(M/N)/F_{d-1}(M/N) \\ &= (F_dM/F_dN)/(F_{d-1}M/F_{d-1}N) = F_dM/F_{d-1}M + F_dN \\ &= (F_dM/F_{d-1}M)/((F_{d-1}M + F_dN)/F_{d-1}M) \\ &= (F_dM/F_{d-1}M)/(F_dN/F_dN \cap F_{d-1}M) = G_d(M)/G_d(N) \end{aligned}$$

We thus have a canonical isomorphism $\phi_d : G_d(M/N) \to (G(M)/G(N))_d$ (of vector spaces or F_0R-modules). We construct the bijection $\phi : G(M/N) \to G(M)/G(N)$, $x = \sum_d x_d \mapsto \sum_d \phi_d(x_d)$. Now it is easy to check that for $r_c \in G(R)_c$ we have that $r_c\phi_d(x_d) = \phi_{c+d}(r_c x_d)$ and that ϕ is a graded morphism of $G(R)$-modules. □

A filtration FM is called **left limited** if $F_nM = 0$ for all $n < n_0$. Of course a left-limited filtration is separated. We say that R is **discretely filtered** (or R is discrete) if FR is left limited, the filtration topology is the discrete topology in this case.

4.2.3 Proposition

Let R be discretely filtered by FR then if $G(R)$ is (left) Noetherian, we have that R is (left) Noetherian.

Proof Look at an *a.c.* $L_0 \subset L_1 \subset \ldots \subset R$ of left ideals of R. Then : $G(L_0) \subset G(L_1) \subset \ldots \subset G(R)$ is an *a.c.* of graded left ideals of $G(R)$ (because the induced filtrations FL_i make the chain $L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \ldots$ into a strict filtered chain). Pick $n_0 \in \mathbb{N}$ such that : $G(L_{n_0}) = G(L_{n_0+1}) = \ldots$, i.e. we have $G(L_{n_0+1}/L_{n_0}) = G(L_{n_0+1})/G(L_{n_0}) = 0$. Since FR is left limited all FL_i are also left limited, hence the gradation of all $G(L_i)$ is left limited. Also $L_{n_{0+1}}/L_{n_0}$ is left limited (therefore separated) so from $G(L_{n_0+1}/L_{n_0}) = 0$ it follows then that $L_{n_0+1}/L_{n_0} = 0$, or $L_{n_0+1} = L_{n_0}$ and the original chain becomes stationary. □

In fact a stronger version of the foregoing proposition holds.

4.2.4 Proposition

If FR is a complete filtration on R then the following are equivalent.

1. $G(R)$ is left Noetherian.
2. R is left Noetherian.
3. $\widetilde{R}$ is left Noetherian.

For the proof we refer to [13], the proposition states in fact that a complete filtration FR with associated graded ring $G(R)$ being (left) Noetherian is a (left) Zariskian filtration.

4.2.5 Corollary

The Weyl algebra $\mathbb{A}_n(K)$ is Noetherian.

Proof With respect to the Bernstein filtration $G(\mathbb{A}_n(K))$ is the polynomial ring $K[X_1, \ldots, X_n]$ which is left and right Noetherian. The Bernstein filtration is positive, hence left-limited so we may apply Proposition 4.2.3. directly. □

We include some results concerning filtered modules over the Weyl algebras, i.e. we shall look at good filtrations.

4.2.6 Definition

Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM. If there exist $m_1, \ldots, m_s$ in M, $k_1, \ldots, k_s \in \mathbb{Z}$, such that for all $n \in \mathbb{Z}$, $F_nM = \sum_{i=s}^{s} F_{n-k_i}Rm_i$, then FM is called a **good filtration** of M.

From this definition it is clear that a module with good filtration is always finitely generated. On the other hand a finitely generated R-module always has good filtrations, indeed for $M \in R$-filt take any $k_1, \ldots, k_s \in \mathbb{Z}$ and put $F_nM = \sum_{i=1}^{s} F_{n-k_i}R.m_i$ if $\{m_1, \ldots, m_s\}$ generates M. However an arbitrary filtration on a finitely generated module need not be good. If FM is good then the quotient filtration induced on M/N is also good for any filtered submodule N with the induced filtrated $FN = N \cap FM$. However FN need not be good at all, for example if R is not left Noetherian and N is a left ideal that is not finitely generated then there is no chance of having $FR \cap N$ a good filtration. Anyway it is clear that finitely generated R-modules may have many good filtrations, however we can control this by the following result.

4.2.7 Proposition

Any two good filtrations on M are equivalent.

Proof Since there exist good filtrations on M, M is finitely generated. For all $n \in \mathbb{Z}$:

$$F_nM = F_{n-d_1}Rm_1 + \ldots + F_{n-d_r}m_r,$$

$$F'_nM = F_{n-e_1}Rm'_1 + \ldots + F_{n-e_s}Rm'_s$$

Choose $w \in \mathbb{Z}$ such that all $m_i \in F'_wM$, and choose $w' \in \mathbb{Z}$ such that all $m'_j \in F_{w'}M$. Put $w'' = \max\{|e_i|, |d_j|, j, i\}$ and $w_0 = |w| + |w'| + |w''|$, then it is easily checked that $F_{n-w_0}M \subset F'_nM \subset F_{n+w_0}M$, proving that the filtrations FM and $F'M$ are indeed (algebraically) equivalent. □

4.2.8 Lemma

Let $M \in R$-filt with filtration FM.

1. FM is good if and only if $\widetilde{M}$ is finitely generated in $\widetilde{R}$-gr.
2. If FM is good then $G(M)$ is a finitely generated $G(R)$-module.

Proof

1. Suppose FM is good, say $F_nM = \sum_{i=1}^{s} F_{n-k_i}Rm_i$, for all $n \in \mathbb{Z}$, where $\{k_1, \ldots, k_s\} \subset \mathbb{Z}$ and $\{m_1, \ldots, m_s\}$ a set of generators for M. By definition of $\widetilde{M}$ we then have $\widetilde{M}_n = \sum_{i=1}^{s} \widetilde{R}_{n-k_i}(m_i)_{k_i}$ where the $(m_i)_{k_i}$ are the homogeneous elements, represented by m_i in $\widetilde{M}_{k_i}$. Hence $\widetilde{M} = \sum_{i=1}^{s} \widetilde{R}(m_i)_{k_i}$. Conversely if $\widetilde{M} = \sum_{i=1}^{i} \widetilde{R}(m_i)_{k_i}$ then we have that $\widetilde{M}_n = \sum_{i=1}^{s} \widetilde{R}_{n-k_i}(m_i)_{k_i}$, for all $n \in \mathbb{Z}$. Then from Proposition 3.2.29 it follows that $F_nM = \sum_{i=1}^{s} F_{n-k_i}Rm_i$, for all $n \in \mathbb{Z}$, hence FM is good.

2. Follows from 1. and Proposition 3.2.29. □

4.2.9 Corollary

Suppose $\widetilde{R}$ is left Noetherian. Then :

1. Good filtrations on M induce good filtrations on submodules.

2. If FM is good and $F'M$ is equivalent to FM then $F'M$ is also good.

3. If FR is also faithful then for $M \in R$-filt with good filtration FM and any submodule $N \subset M$ we have : $N = \cap_{n\in\mathbb{Z}}(N + F_nM)$, that is : N is closed in the filtration topology of M, moreover any good filtration is separated. If $G(M)$ may be generated by n homogeneous elements then M may be generated by n generators. Moreover, if $N \subset P \subset M$ are submodules with induced filtration by FM, then $G(N) = G(P)$ implies $N = P$

Proof Since $\widetilde{R}$ is left Noetherian, statements 1. and 2. follow easily from Lemma 4.1.8. Recall Theorem 4.1.15., so in proving 3. we have that good filtrations are separated. The quotient filtration $F(M/N)$ is still good hence separated, hence $(\cap_{n\in\mathbb{Z}}(N+F_kM))/N = \cap_{n\in\mathbb{Z}}F_k(M/N) = 0$ yields that N is closed as claimed.

Assume $G(M) = \sum_{i-1}^{s} G(R)\sigma(u_i)$ where $u_i \in F_{k_i}M - F_{k_{i-1}}M$, $\sigma(u_i) = u_i \text{mod } F_{k_i-1}M$.

Then for each $n \in \mathbb{Z}$, we have : $G(M)_n = \sum_{i=1}^{s} G(R)_{n-k_i}\sigma(u_i)$,
$F_nM = \sum_{i=1}^{s} F_{n-k_i}Ru_i + F_{n-1}M$.

Thus, it follows that $M \subset \cap_{k\in\mathbb{Z}}(\sum_{i=1}^{s} Ru_i + F_kM) = \sum_{i=1}^{s} Ru_i$, since the latter is closed as observed before. Finally since P/N has good filtration (hence separated) then $G(P/N) = G(P)/G(N) = 0$ entails $P/N = 0$. □

With some more work but no essentially new ingredients we may establish the following result.

4.2.10 Proposition

Let FR be complete and $G(R)$ left Noetherian and FM separated then FM is good if and only if $G(M)$ is finitely generated. Moreover, if $G(M)$ is generated by s homogeneous elements as a $G(R)$-module then M can be generated by s (or less) elements as an R-module.

Proof (cfr. [13, Theorem 5.7.] in fact again it follows from the fact that a complete FR with left Noetherian $G(R)$ is a left Zariskian filtration (in particular $\widetilde{R}$ is left Noetherian and FR is faithfull) then this theorem follows from the foregoing. □

For $R = \mathbb{A}_1(K)$ (similar for $\mathbb{A}_n(K)$) we have that $\widetilde{R}$ is Noetherian and positively graded, $F_{-1}R = 0 \subset J(F_0R)$ also holds. Therefore Corollary 4.2.9. holds for $R = \mathbb{A}_1(K)$, hence submodules of modules with good filtration have an induced good filtration and they may be generated by (less than) d elements if the associated graded module may be generated by d elements over $G(\mathbb{A}_1(K)) \cong K[X,Y]$.

These results may be extended to other rings of differential operators and to other positively filtered rings having left Noetherian associated graded rings like twisted polynomial rings (cf. Witten algebras and generalized gauge algebras in [21]). Let us finish this section with a result on finite dimensional (over K) modules over $\mathbb{A}_1(K)$.

4.2.11 Proposition

Let M be a left $\mathbb{A}_1(K)$-module, then if $\dim_K M$ is finite we must have $M = 0$, i.e. the Weyl algebra does not have finite dimensional representations.

4.2.12 Proof

Suppose V has $\dim_K V = n$ and V is a left $\mathbb{A}_1(K)$-module. The action of $x \in \mathbb{A}_1(K)$ on V is given by an $n \times n$-matrix μ_x, to the action of y there corresponds a matrix μ_y. We know that $tr(\mu_y\mu_x - \mu_x\mu_y) = 0$ but from $yx - xy = 1$ we have that $tr(\mu_y\mu_x - \mu_x\mu_y) = trI = n$. Thus $n = 0$, i.e. there does not exist such a nontrivial V. □

There are of course left modules over the Weyl algebra $\mathbb{A}_1(K)$ that by restriction of scalars are finitely generated modules over $K[x]$, in fact $K[x]$ itself is such a module where x acts by multiplication and y by $\frac{\partial}{\partial x}$.

Module theory is usually built on the simple or irreducible modules and direct sums of finitely many of these. This is the basis of classical representation thenory of finite groups by introducing the complete reducibility of representation related to semisimplicity of modules. In the case of the Weyl algebras a classification of simple left modules for $\mathbb{A}_1(\mathbb{C})$ was obtained by J. Beck and for $\mathbb{A}_2(\mathbb{C})$ by Bavula, Van Oystaeyen [1]. It is open for higher Weyl algebras. We will not go deeper into this here.

4.3 Semisimple Rings and Modules

Let R be a associative ring with unit, M a right R-module. Put $A(M) = \{x \in R, Mx = 0\}$. We say that M is **faithful** if $A(M) = 0$. It is clear that $A(M)$ is an ideal of R and M is a faithful $R/A(M)$-module. Recall that M is said to be a simple module if O and M are the only R-submodules of M. Let $E(M)$ be the set of additive endomorphisms of M. Then $E(M)$ is a ring and the map $\phi : R \to E(M), a \mapsto t_a$, $t_a(m) = ma$ for all $m \in M$, is a ring homomorphism with $\mathrm{Ker}\phi = A(M)$. Thus $R/A(M)$ is isomorphic to a subring of $E(M)$.

4.3.1 Proposition

If M is a simple R-module, then $C(M) = \{\Psi \in E(M), t_a\Psi = \Psi t_a \text{ for all } a \in R\}$ is a skewfield.

Proof Suppose $\Theta \neq 0$ is in $C(M)$. If $\Theta^{-1} \subset E(M)$ then $\Theta^{-1} \in C(M)$, this is obvious. Since $\Theta \neq 0$ we have $W = M\Theta \neq 0$, we write the action of Θ in $C(M)$ on the right. For $r \in R$, $Wr = Wt_r = M\Theta t_r = Mt_r\Theta \subset M\Theta = W$. Thus $M\Theta$ is an R-module (on the right), hence $M\Theta \neq 0$ yields $M\Theta = M$ as M is simple and Θ is therefore a surjective map. Since KerΘ is also a right R-submodule of M we must have Ker$\Theta = 0$ or Θ is also injective. Since Θ is a right R-linear bijection, $\Theta^{-1} \in E(M)$. $\square$

4.3.2 Lemma

A simple right R-module M is isomorphic to R/ρ for some maximal right ideal ρ of R. Moreover, there is an $a \in R$ such that $x - ax \in \rho$ for all $x \in R$ (even if R were a ring without unit). For any maximal right ideal ρ of R, R/ρ is a simple right R-module.

Proof If $m \neq 0$ in M then $\pi : R \to M, 1 \mapsto m$ is right R-linear. As M is simple $M = mR$ and thus $M = R/\rho$ where $\rho = A(m)$. A right ideal of R containing ρ is mapped by π on a submodule of M, hence ρ is maximal. The other statements are obvious. $\square$

4.3.3 Definition

The **Jacobson radical** $J(R)$ of R is the set $\{r \in R, Mr = 0$ for all simple right R-modules $M\} = \cap\{A(M), M$ simple right R-module$\}$. Hence $J(R)$ is an ideal of R.

If ρ is a maximal right ideal of R and $M = R/\rho$, define $(\rho : R) = \{x \in R, Rx \subset \rho\}$. The latter is an ideal of R and one sees directly that $A(R/\rho) = (\rho : R)$. Indeed if $x \in A(M)$, then $Mx = 0$ i.e. $Rx \subset \rho$, conversely $Rx \subset \rho$ means that x annihilates R/ρ. It follows from this that :

$$
\begin{aligned}
J(R) &= \cap\{\rho, \rho \text{ a maximal right ideal of } R\} \\
&= \cap\{(\rho : R), \rho \text{ a maximal right ideal of } R\}
\end{aligned}
$$

Consider $x \in J(R)$ and look at $S = \{xy + y, y \in R\}$. If $S \neq R$ then there is a maximal right ideal $\rho_0 \supset S$. Since $x \in \rho_0$ it follows that $xy \in \rho_0$, what leads to $y \in \rho_0$; the latter holds for all $y \in R$, hence we must have that $R = \{xy+y, y \in R\}$. In particular, there is a $w \in R$ such that : $-x = xw+w$, i.e. $x+w+xw = 0$. An element $a \in R$ is called a **right quasi-regular element** if there is an $a' \in R$ such that $a + a' + aa' = 0$. A right ideal of R is **(right) quasi-regular** if every element of it is right quasi regular.

4.3.4 Exercise

$J(R)$ contains every right quasi-regular right ideal of R. This also holds for the left symmetric version. The left Jacobson radical $J_l(R)$ is also an ideal of R. The exercise entails that $J_l(R) = J(R)$.

A right ideal of R is **nil** if every element of it is a nilpotent element. A right ideal of R is **nilpotent** if there is an $m \in \mathbb{N}$, $m \neq 0$, such that $a_1 \dots a_m = 0$ for all $a_1, \dots, a_m \subset \rho$, i.e. $\rho^m = 0$ for some nonzero $m \in \mathbb{N}$.

4.3.5 Lemma

Every nil right (or left) ideal of R is contained in $J(R)$.

Proof If $a^m = 0$ for $a \in R$, put $b = -a + a^2 - a^3 + \dots (-)^{m-1} a^{m-1}$ and calculate $a + b + ab = 0$, $\square$

4.3.6 Proposition

We have $J(R/J(R)) = 0$.

Proof Let $\pi : R \to \overline{R} = R/J(R)$ be the canonical epimorphism. If ρ is a maximal right ideal of R, then $\rho \supset J(R)$ and $\overline{\rho} = \rho/J(R)$ is a maximal right ideal of $\overline{R}$. Since, $J(R) = \cap\{\rho$, ρ a maximal right ideal of $R\}$, we obtain : $0 = \cap\{\overline{\rho}, \rho$ a maximal right ideal of $R\}$, then $J(\overline{R}) \subset 0$, or $J(\overline{R}) = 0$. $\square$

4.3.7 Proposition

If the ring R is right Artinian (similarly, left Artinian), then $J(R)$ is nilpotent.

Proof Look at the descending chain : $J \supset J^2 \supset \dots \supset J^n \dots$. For some $n \in \mathbb{N}$, $J^n = J^{n+1} = \dots = J^{2n} = \dots$, hence $xJ^{2n} = 0$ if and only if $xJ^n = 0$. Put $W = \{x \in J, xJ^n = 0\}$; this is clearly an ideal of R. If $W \supset J^n$ then $J^n.J^n = 0$ yields $J^n = 0$ what we desire to establish. So assume $W \not\supset J^n$ and look at $\overline{R} = R/W$, where $\overline{J}^n \neq 0$. If $\overline{x}.\overline{J^n} = 0$ then we have $xJ^n \subset W$ i.e. $xJ^{2n} = 0$ hence $xJ^n = 0$, or $x \in W$ and $\overline{x} = 0$. Hence we established that $\overline{x}\overline{J^n} = 0$ implies $\overline{x} = 0$. As $\overline{R}$ is also right Artinian, $\overline{J^n}$ contains a minimal right ideal $\overline{\rho} \neq 0$ for $\overline{R}$. Then $\overline{\rho}$ is a simple $\overline{R}$-module and as such it is annihilated by $J(\overline{R})$. However, from $\overline{J^n} \subset J(\overline{R})$ then $\overline{\rho}\overline{J^n} = 0$ follows; the latter entails $\overline{\rho} = 0$ by the above argument. So we arrive at a contradiction hence $J^n = 0$. $\square$

4.3.8 Corollary

In a right Artinian ring R every nil right ideal is nilpotent.

Proof If ρ is a nil right ideal then $\rho \subset J(R)$ and $J(R)$ is nilpotent, hence ρ is nilpotent. $\square$

A ring R is **semisimple** if $J(R) = 0$ and it is semisimple (left) Artinian if it is semisimple plus left Artinian (similar with right Artinian). Finite direct sums of matrix rings over skewfields are semisimple Artinian (left and right) rings. These rings appear naturally in the representation theory of finite groups, we refer to M. Hall's classic book, [8], for this theory but provide a very short introduction to it here.

Let G be a finite group; we may view G as acting on itself via multiplication (on the right) and represent $g \in G$ by the permutation matrix, i.e. the $|G| \times |G|$-matrix associated to $h \mapsto hg$ for all $h \in G$. The permutation group on $|G|$-elements has $|G|!$ elements and this grows very big quickly. Another way of representing G is be viewing it as a group of operators on some vector space over some field F. The group algebra of G over F, denoted by FG, is the vector space $FG = \{\sum_i \alpha_i g_i, \alpha_i \in F, g_i \in G\}$ with multiplication defined on it defined as the bilinear extension of the group multiplication, that is :

$$(\sum_{g_i \in G} \alpha_i g_i)(\sum_{f_j \in G} \beta_j f_j) = \sum_{h \in G}(\sum_{h = g_i f_j} \alpha_i \beta_j)h$$

If G acts on an F-vectorspace V by automorphisms, i.e. there is given a groupmorphism $G \to \mathrm{Aut}_K V$, $g \mapsto (v \mapsto g(v))$, then V is a right FG-module by putting : $v(\sum_{g_i \in G} \alpha_i g_i) = \sum_{g_i \in G} \alpha_i g_i(v)$. We have to find F-vectorspaces with G-action, or equivalently FG-modules. The key to the technique of representation theory for finite groups is Maschke's theorem.

4.3.9 Theorem (Maschke)

Let G be a finite group of order n and let F a field with $(\mathrm{char}F, n) = 1$ or $\mathrm{char}F = 0$. The ring FG is semisimple Artinian.

Proof Since $[FG : F] < \infty, FG$ is a (left and right) Artinian ring. Pick $a \in FG$ and $t_a : FG \to FG$, $x \mapsto xa$, this defines the right regular representation of FG (by acting on itself). Clearly t_a is F-linear and $t : a \mapsto t_a$ defines an algebra morphism of FG onto an algebra of F-linear transformations on the F-vector space FG. We use the elements of G as a basis for FG over F and write t_a as an $n \times n$-matrix over F in the fixed basis. We have $\mathrm{tr}(t_1) = n$ and $\mathrm{tr}(t_g) = 0$ for $g \neq 1$ in G. Put $J = J(FG)$; we know that J is a nilpotent ideal. Suppose $0 \neq x = \alpha_1 g_1 + \ldots + \alpha_n g_n \in J$. By multiplying with g_1^{-1} (note that J is an ideal of FG) we may assume that $0 \neq x = \alpha_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n$ with $\alpha_1 \neq 0$. Since t_x is nilpotent we have $\mathrm{tr}(t_x) = 0$ (follows from the Cayley-Hamilton theorem for matrices for example). On the other hand $\mathrm{tr}(t_x) = \alpha_1 \mathrm{tr}(t_1) + \ldots + \alpha_n \mathrm{tr}(t_{g_n}) = \alpha_1 n$ and thus $0 = \alpha_1 n$. From the assumption that either F has characteristic zero, or characteristic prime to n, it then follows that $\alpha_1 = 0$, a contradiction. Consequently $J(FG) = 0$ or FG is semisimple Artinian. $\square$

4.3.10 Proposition

Let R be a semisimple (right) Artinian ring and $\rho \neq 0$ a right ideal of R, then there exists an idempotent e in R, i.e. $e^2 = e$, such that $\rho = eR$.

Proof Since $\rho \neq 0$ it is not nilpotent, hence $\rho^2 \neq 0$. Hence there is an $x \in \rho$ such that $x\rho \neq 0$ and $x\rho \subset \rho$. Suppose ρ is a minimal right ideal of R, then $x\rho = \rho$ follows and thus there is an $e \in \rho$ such that $xe = x$. From $xe^2 = xe = x$, it follows then that $x(e^2 - e) = 0$. Put $\rho_0 = \{a \in \rho, xa = 0\}$, this is a right ideal contained in ρ and $\rho_0 \neq \rho$ since $x\rho \neq 0$; the assumption on ρ thus leads to $\rho_0 = 0$ and as $e^2 - e \in \rho_0$ we arrive at $e^2 = e$ with $e \neq 0$ because $xe = x \neq 0$. From $0 \neq eR \subset \rho$ then follows $\rho = eR$ as desired. Now let ρ be arbitrary, the ρ contains a minimal right ideal ρ' because R is right Artinian; then ρ also contains an idempotent e (generator of ρ'). Put $A(e) = \{x \in \rho, ex = 0\}$. The set $\{A(e), e^2 = e \neq 0 \in \rho\}$ is nonempty and thus it has a minimal element, say $A(e_0)$.

If $A(e_0) = 0$ then from $e_0(x - e_0x) = 0$, for all $x \in \rho$, it follows that $x - e_0x \in A(e_0) = 0$, or $x = e_0x$ for all $x \in \rho$. Thus $\rho = e_0\rho \subset e_0R \subset \rho$ and this leads to $\rho = e_0R$. So suppose $A(e_0) \neq 0$; then there is an idempotent $e_1 \in A(e_0)$. From the definition of $A(e_0)$ it follows that $e_1 \in \rho$ and $e_0e_1 = 0$. Put $e^* = e_0 + e_1 - e_1e_0 \in \rho$. Calculate $(e^*)^2 = e^*$ and $e^*e_1 = (e_0 + e_1 - e_1e_0)e_1 = e_1 \neq 0$ thus also $e^* \neq 0$. If $e^*x = 0$ then $(e_0 + e_1 - e_1e_0)x = 0$ and thus $e_0(e_0 + e_1 - e_1e_0)x = 0$, or $e_0x = 0$. All of this means that $A(e^*) \subset A(e_0)$, $e_1 \in A(e_0) - A(e^*)$. The minimality of $A(e_0)$ yields a contradiction, hence $A(e_0) \neq 0$ is excluded. This establishes the claims. □

4.3.11 Remark

The above is for onesided ideals but it also implies a structure result for (two-sided) ideals.

4.3.12 Corollary

If I is an ideal of a semisimple (right)Artininan ring R, then $I = Re = eR$ for an idempotent e in the centre $Z(R)$ of R.

4.3.13 Corollary

A semisimple right Artinian ring R is a finite direct sum of simple right Artinian rings S_i, $R = \oplus_{i=1}^{d} S_i$ where S_i is a simple right Artininan ring with unit e_i, $S_i = e_iR$.

Proof Look at minimal ideals $S_i, i = 1, \ldots, d$, with corresponding central idempotents. It is clear that $\sum_{i=1}^{d} S_i = \oplus_{i=1}^{d} S_i$ and $R = \oplus_{i=1}^{d} S_i$ follows because otherwise $1 - \sum e_i$ is an idempotent and $R(1 - \sum_{i=1}^{d} e_i)$ contains a minimal ideal of R i.e. one of the S_i. All statements follow easily. □

4.3.14 Remark

If R is simple (right) Artinian then $Z(R)$ is a field.

Proof Let $x \neq 0$ in $Z(R)$. Then $Rx = xR$ is an ideal of R thus $Rx \neq 0$ entails $Rx = R$. Hence there is a $y \in R$ such that $yx = 1$. In a similar way there is a $z \in R$ such that $xz = 1$. Then $y = yxz = (yx)z = z$ and x has an inverse in R. For any $a \in R$ we have $x(ay - ya) = xay - a = axy - a = a - a = 0$ and thus $yx(ay - ya) = 0$ or $ay - ya = 0$. Thus $y \in Z(R)$, it follows that every $x \neq 0$ in $Z(R)$ has an inverse in $Z(R)$. □

4.3.15 Exercises

1. If Δ is a skewfield then $M_n(\Delta)$ is a simple Artinian ring.

2. For an arbitrary ring R the ideals of $M_n(R)$ are exactly the $M_n(I)$ where I is an ideal of R (Hint : use elementary operations with matrix units $e_{ij}, i, j = 1, \ldots, n$).

3. In a right Artininan ring, every prime ideal is maximal, or equivalently a right Artinian prime ring is necessarily a simple ring (Hint : The centre of a right Artinian prime ring is a domain and thus contains no idempotents except 0 and 1).

4.3.16 Definition

A ring R is primitive if there is a faithful simple right R-module. In fact this should be called right primitive; it is not equivalent with left primitive (there is a counter-example due to G. Bergman).

If ρ is a maximal right R-ideal then $A(M) = (\rho : R)$ if $M = R/\rho$, thus $R/(\rho : R)$ is a primitive ring. It is clear that R is primitive if and only if there is a maximal right ideal ρ of R such that $(\rho : R) = 0$,. i.e. ρ does not contain a proper ideal.

4.3.17 Theorem

A primitive ring R with faithful simple module M is a dense ring of linear transformations of M over $C(M)$, in other words : for every n and $v_1, \ldots, v_n$ in M which are $C(M)$-linear independent there exists an $r \in R$ such that $v_i r$ equals an element $w_i \in M$ which is beforehand fixed, $i = 1, \ldots, n$.

Proof Somewhat thechnical. We refer to the classical books. N. Jacobson [11], I. Herstein, [9]. □

4.3.18 Theorem (Wedderburn, Artin)

If R is a simple (right) Artinian ring then $R \cong M_n(D)$ for some $n \in \mathbb{N}$ and some skewfield D. Both n and the isomorphism class of D are uniquely determined by R.

Proof It is obvious that R is primitive. Let M be a faithful simple right module. The commuting ring $C(M) = D$ is a skewfield and we may consider M as D-vector space. Let $\{v_1, \ldots, v_m\}$ be D-independent in M. Define : $\rho_m = \{x \in R, v_i x = 0 \text{ for } i = 1, \ldots, m\}$. This yields a descending chain of right ideals of R : $\rho_1 \supset \rho_2 \supset \ldots \supset \rho_m \supset \ldots$. From the foregoing theorem it follows that the sequence is strictly descending and thus $\rho_n = 0$ for some $n \in \mathbb{N}$. If v_{n+1} is not D-linearly dependent on $\{v_1, \ldots, v_n\}$ then by the density in foregoing theorem there exists an $r \in R$ such that $v_i r = 0$ for $i = 1, \ldots, n$ and $v_{n+1} r = 1$; this would contradict $\rho_n = 0$ hence it follows that M is finite dimensional over D. Now, again by the density argument we obtain that R is the ring of all D-linear maps $M \to M$, that is $R = M_n(D)$, $n = \dim_d M$. Unicity of n and the isomorphism class of D will follow if we can establish that $M_m(\Delta) = M_n(D)$ if and only if $m = n$ and $D \cong \Delta$. Let $\phi : M_n(D) \to M_m(\Delta)$ be an algebra isomorphism and put $f = \phi(e_{11}), e_{11} \in M_n(D)$. Since $e_{11} M_n(D)$ is a minimal right ideal of $M_n(D)$, $f M_m(\Delta)$ must be a minimal right ideal of $M_m(\Delta)$. By change of basis, i.e. applying an automorphism of $M_m(\Delta)$ we may write f in the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ unit matrix. Since $fM_m(\Delta)$ is minimal we must have $r = 1$, hence without loss of generality we may assume that $f = f_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now we obtain : $D \cong e_{11}M_n(D)e_{11} \cong f_{11}M_m(\Delta)f_{11} \cong \Delta$. Moreover, $e_{11}M_n(D)$ is an n-dimensional D-space, whereas $f_{11}M_m(\Delta)$ is an m-dimensional $\Delta = D$-space. Hence $n = m$ also follows. □

4.3.19 Corollary

Combination of foregoing results yields that a semisimple (left or right) Artinian ring is a direct sum of matrix rings over skewfields. Hence the right Artinian case coincides with the left Artinian.

4.3.20 Proposition

Let K be an algebraically closed field, then a skewfield D which is central and algebraic over K is necessarily equal to K.

Proof Any $a \neq 0$ in D is a solution for some $p(X) \in K[X]$. Since K is algebraically closed, $p(X) = \prod_{i=1}^{n}(X - \lambda_i)$ for some $\lambda_i \in K$. Thus $(a - \lambda_1)\ldots(a - \lambda_n) = 0$ in D, or $a = \lambda_i$ for some λ_i and $a \in K$. It follows that $D = K$. □

4.3.21 Corollary

Let G be a finite group, K an algebraically closed field with $\mathrm{char}(K) = 0$. Then $KG = M_{m_1}(K) \oplus \ldots \oplus M_{n_d}(K)$.

Proof Directly from Maschke's theorem and the foregoing proposition. □

One could ask about the existence of finite skewfields ? It turns out these are always commutative.

4.3.22 Theorem (Wedderburn)

A finite skewfield is commutative.

Proof Let D be a skewfield with centre $Z(D) = K$. If D is finite then $\mathrm{char}(D) = p \neq 0$; suppose D has $q = p^n$ elements. If $n = 1$ then $D = \mathbb{Z}/p\mathbb{Z}$ and then the theorem holds, hence we now argue by induction, assuming that skewfields with less than q elements are commutative. If $a, b \in D$ are such that $ab \neq ba$ but $b^t a = ab^t$ for some $t \in \mathbb{N}$ then we look at $N(b^t) = \{x \in D, xb^t = b^t x\}$. Clearly $N(b^t)$ is a subfield of D which is noncommutative because $a, b \in N(b^t)$. By the induction hypothesis we must have $N(b^t) = D$, hence $b^t \in Z(D) = K$. For $n \in D$ let $m(u)$ be the smallest positive number such that $u^{m(u)} \in K$. Choose $a \in D - K$ such that $r = m(a)$ is minimal amongst the $m(y), y \in D$; then obviously r is a prime number. We now establish a sublemma.

Sublemma Let D be a skewfield, $\mathrm{char}(D) = p \neq 0$, and $Z(D) = K$. If $a \in D - K$ is such that $a^{p^n} = a$ for certain $n \geq 1$ then there is an $x \in D$ such that there is an $i \in \mathbb{Z}$ such that $xax^{-1} = a^i \neq a$.

Proof of the Sublemma Define $\delta : D \to D, x \mapsto xa - ax$. One calculates : $x\delta^{p^k} = xa^{p^k} - a^{p^k}x$, for all $k \geq 0$, (action of δ is written on the right). Now $k = \mathbb{Z}/p\mathbb{Z}[a]$ is a finite field with p^n elements i.e. $a^{p^n} = a$ and thus $x\delta^{p^n} = xa^{p^n} - a^{p^n}x = xa - ax = x\delta$, or $\delta^{p^k} = \delta$. If $\lambda \in K$ then : $(\lambda x)\delta = (\lambda x)a - a(\lambda x) = \lambda(xa - ax) = \lambda(x\delta)$, or left multiplication by $\lambda \in k$ commutes with δ.

The polynomial $X^{p^n} - X$ decomposes over k in a product of linear factors : $\prod_{\lambda \in k}(X - \lambda)$. Since $m(\lambda)$ commutes with δ, as observed above, we obtain : $0 = \delta^{p^n} - \delta = \prod_{\lambda \in k}(\delta - \lambda)$. Since $a \notin K$ we have $\delta \neq 0$.

Consider a shortest product : $0 = \delta(\delta - \lambda_1)\dots(\delta - \lambda_l), \lambda_i \in k$, it is clear that $l \geq 1$. For some $z \neq 0$ in D we have : $z(\delta(\delta - \lambda_1)\dots(\delta - \lambda_{l-1})) = w \neq 0$ but $w(\delta - \lambda_l) = 0$. The last equality means : $wa - aw = \lambda_l w$, $\lambda_l \neq 0$ in k. From $w \neq 0$ it follows : $waw^{-1} = \lambda_l + a \neq a$ but it is in k. Now $waw^{-1} \in k$ has the same order as a, hence $waw^{-1} = a^i$ for some $i \in \mathbb{N}$.

Back to the proof of the theorem now. We have an $x \in D$ such that $xax^{-1} = a^i \neq a$, hence $x^k ax^{-k} = a^{i^k}$ and for $l = r - 1$ we obtain (because $i^{r-1} = 1$ modulo r) : $x^{r-1}ax^{i-r} = \lambda a$ for some $\lambda \in k$. Since $xa \neq ax$ and $x^{r-1} \notin K$ we must have $\lambda \neq 1$. Put $b = x^{r-1}$, then $bab^{-1} = \lambda a$, $\lambda^r a^r = (bab^{-1})^r = ba^r b^{-1} = a^r$ (as $r = m(a)$). It follows that $\lambda^r = 1$ and then $b^r ab^{-r} = a$; this entails $b^r \in K$, say $a^r = \alpha, b^r = \beta \in K$.

Claim If $u + u_1 b + \dots + u^{r-1}b^{r-1} = 0$ with $u_i \in K(a)$ then $u_i = 0$ for $i = 1, \dots, r-1$. Indeed, suppose

$$(*) \qquad u_0 + u_1, b^{m_1} + \dots + u_{m_l}b^{m_l} = 0$$

is a shortest relation with $r > m_l > \dots > m_1$. Conjugating by a and using $a^{-1}b^i a = \lambda^i b^i$, we obtain the following :

$$(**) \qquad u_0 + \lambda^{m_1}u_1 b^{m_1} + \dots + \lambda^{m_l}u_l b^{m_l} = 0$$

Since r is prime, $\lambda^r = 1$, but $\lambda \neq 1$, we may combine (*) and (**) to obtain a shorter relation, contradiction. In particular, it follows from the claim that $X^r - \alpha$, resp. $X^r - \beta$ are the minimal polynomials of a and b over K, hence $[K(a) : K] = [K(b) : K] = r$. We induce ϕ on $K(a)$ by $\phi(\lambda) = b\lambda b^{-1}$ for $\lambda \in K(a)$. Powers of ϕ yield all (exactly r) K-automorphisms of $K(a)$.

Since $K(a)$ is finite every element of K is a norm of an element of $K(a)$, i.e. $\lambda \in K$, then $\lambda = x\phi(x)\dots\phi^{r-1}(x)$ for some $x \in K(a)$. In particular $\beta^{-1} = y\phi(y)\dots\phi^{r-1}(y)$ for some $y \in K(a)$. However we have : $(1 - yb)(1 + yb + y\phi(y)b^2 + \dots + y\phi(y)\dots\phi^{r-2}(y)b^{p-1}) = 0$. This yields that either $1 - yb = 0$ or $(1 + yb + \dots) = 0$. Since $y \in K(a)$ it follows from the claim that each of these equalities is unacceptable, contradiction. Hence we must have $m(a) = 1$ for all $a \in D$, hence D is commutative. $\square$

4.3.23 Corollary

Let D be a skewfield with $\mathrm{char}(D) = p \neq 0$. If G is a finite multiplicative subgroup of $D^* = D - \{0\}$, then G is abelian and therefore cyclic.

Proof Let $\mathbb{F}_p$ be the prime field in D, then $F_P G \subset D$. Since $F_P G$ is Artinian and a domain it follows that $F_P G$ is a skewfield (from the Artin - Wedderburn theorem). Since $F_P G$ is a finite ring it follows from Wedderburn's theorem that $F_P G$ is commutative hence G is abelian. A finite subgroup of a finite field is cyclic even ! □

A ring is **simple** if it has no nontrivial ideals. The centre of a simple ring is always a field. If R is finite dimensional over $K = Z(R)$ then R is right and left Artinian but the converse need not hold as any skewfield of infinite dimension over its center shows.

A simple ring need not be a domain, for example $M_n(\Delta)$, where Δ is a skewfield is simple but not a domain unless $n = 1$. Do there exist simple domains that are not skewfields ? Yes, in fact we arrive at an example of this by looking at the Weyl algebra.

4.3.24 Theorem

The Weyl algebra $\mathbb{A}_1(K)$, where $\mathrm{char} K = 0$, is a Noetherian simple domain.

Proof Suppose that $J \neq 0$ is an ideal of $\mathbb{A}_1(K)$ and $z \neq 0$ in J, say $z = p_0 + p_1 y + \ldots + p_s y^s$, $p_i \in K[x]$ and $p_s \neq 0$. Use the relation $y^i x - x y^i = i y^{i-1}$ to arrive at : $zx - xz = p_1 + 2p_2 y + \ldots + s p_s y^{s-1}$. Since J is an ideal and $z \in J$ we have $xz - zx \in J$. If $s > 1$ then we continue with $(zx - xz)x - x(zx - xz)$, and so on. After at most s steps we obtain that J contains $s! p_s \neq 0$. Rewrite $s! p_s$ as $\lambda_0 + \lambda_1 x + \ldots + \lambda_m x^m = p$ with $\lambda_i \in K, \lambda_m \neq 0$. Calculate $yp - py$, it is again an element of J : $yp - py = \lambda_1 + 2\lambda_2 x + \ldots + m\lambda_m x^{m-1}$. Repeat this at most m-times until we reach : $m!\lambda_m \in J$. But $m!\lambda_m$ is in K and $m!\lambda_m \neq 0$, contradiction to $J \cap K = 0$. It follows that $J = \mathbb{A}_1(K)$ or $\mathbb{A}_1(K)$ is simple. □

4.3.25 Remark

For K with $\mathrm{char}(K) = 0$, $\mathbb{A}_n(K)$ is a simple Noetherian domain too; the proof is similar to the case $n = 1$. Also $\mathbb{A}_n(K)$ is not Artinian and not a skewfield, moreover $\mathbb{A}_n(K)$ has no finite dimensional (over K) modules.

We may view the Weyl algebra $\mathbb{A}_n(K)$ as a ring of linear transformation on $K[x_1, \ldots, x_n]$.

For a left R-module M we look at $R \to \mathrm{End}_+(M)$ by associating to $r \in R$ the additive endomorphism of M, $\mu_r : M \to M, m \mapsto rm$. The centralizer of $\{\mu_r, r \in R\}$ in $\mathrm{End}_T(M)$ is exactly $\mathrm{End}_R(M) = S$; let T be the centralizer of S where M is regarded as an (S, R)-bimodule. We call T the **bicentralizer** of R on M. We say that R acts **densely** on M if for all $n \in \mathbb{N}$ and for all x in $M^{(n)} = M \oplus \ldots \oplus M$, the sum ranging over n copies of M, for every $\Theta \in T \subset \mathrm{End}_T(M)$, where T is the bicentralizer of R on M, there exists an $a \in R$ such that $\Theta(x) = ax$.

4.3.26 Theorem (Form of the density theorem)

Every simple ring is a dense ring of linear transformations over a skewfield.

The theory of simple Noetherian rings is much less developed than its Artinian counterpart. There is the book of C. Faith, J. Cozzens, [4] but the results there do not learn much about the Weyl algebra.

Chapter 5

Localization at Ore Sets

Let A be a ring, as always associative and with unit 1. An $a \in A$ is **right regular** if $za = 0$ with $z \in A$ entails $z = 0$. We say that $a \in A$ is **regular** if it is right and left regular. A **multiplicative** set S in A is a subset such that $1 \in S$ and if $s_1, s_2 \in S$ then $s_1 s_2 \in S$; we always assume $0 \notin S$. A **left ring of fractions of A with denominator set S** is an overring Q of A such that every element $s \in S$ is invertible in Q and $Q = S^{-1}A$ (since Q is an overring it follows that S cannot contain zero-divisors of A, indeed if $t \in S$ is such that $ta = 0$, then $t^{-1}(ta) = 0$ in Q and $a = 0$ follows).

5.1. Proposition (Ore)

There is a left ring of fractions of A with denominator set S if and only if S does not contain zerodivisors of A and S satisfies the left Ore condition : for $s \in S, a \in A$ we have $As \cap Sa \neq \emptyset$, that is there exist $a' \in A, s' \in S$, such that $s'a = a's$.

Proof Assume $Q = S^{-1}A$ exists. We aready observed that S cannot contain zerodivisors of A then. For $a \in A$, $s \in S$ we have $as^{-1} \in Q$ thus there is an $s' \in S$ such that $s'(as^{-1})$ is in A, say $a' = s'as^{-1}$. Hence $s'a = a's$ or the left Ore condition holds. Conversely if S satisfies the left Ore condition, then we construct Q by defining on $S \times A$ a relation $\sim$ by $(s, a) \sim (t, b)$ if and only if there exist $x, y \in A$ such that $xs = yt \in S$ and $xa = yb$. It is clear that $\sim$ is an equivalence relation. Put $Q = (S \times A)/\sim$ and write $s^{-1}a$ for the class of (s, a). Two elements of Q, say $s^{-1}a$ and $t^{-1}b$ may be written with a same denominator by the left Ore condition. The sum in Q is defined by $s^{-1}a + s^{-1}b = s^{-1}(a + b)$ and the product $(h^{-1}a)(t^{-1}c)$ for $h, t \in S$ and $a, c \in A$, is defined by $(sh)^{-1}bc$ where $s \in S, b \in A$ are such that $at^{-1} = s^{-1}b$. Mapping a to the class of $(1, a)$ identifies A as a subring of Q. $\square$

From the construction of $Q = S^{-1}A$ in the foregoing proof we derive the folowing universal property: for every ring homomorphism $\varphi : A \to B$ such that $\varphi(S)$ is inversible in B there is a unique ring homomorphism $\psi : S^{-1}A \to B$ making the following diagram commutative :

$$\begin{tikzcd}
A \arrow[rr, "j_S"'] \arrow[dr, "\varphi"'] & & S^{-1}A \arrow[dl, "\psi"] \\
& B &
\end{tikzcd}$$

where j_S is the canonical $A \to S^{-1}A, a \mapsto (1,a)/\sim$. We may interchange left and right in foregoing definitions and obtain the **right ring of fractions** AS^{-1} if the right Ore condition holds ($sA \cap aS \neq \emptyset$). If S satisfies both the left and right Ore conditions then $S^{-1}A$ and AS^{-1} both exist; using the universal property of both $S^{-1}A$ and AS^{-1} we may conclude that we may identify $S^{-1}A$ and AS^{-1}.

5.2. Examples

1. Let s be a regular element of A and $S = \{1, s, s^2, \ldots\}$. If $sA \subset As$ then S is a left Ore set in A (i.e. the left Ore condition holds).

2. Let A be a Noetherian domain and $S = A - \{0\}$. Then S is a left and right Ore set in A and $S^{-1}A$ is a skewfield.

   Proof Look at $s \in S, a \in A$ and the ascending chain of left ideals of A : $Aa + Aas + \ldots + Aas^n, n \in \mathbb{N}$. For large n we obtain from the a.c.c. on A that :

   $as^{n+1} = a_0a + a_1as + \ldots + a_nas^n$, for some $a_i \in A, a_0 \neq 0$ (choose n minimal and use that s is not a zero divisor), thus : $(as^n - a_nas^{n-1} + \ldots + \ldots - a_1a)s = a_0a \in As \cap Sa$. □

   Particular case : $A = \mathbb{A}_1(K)$ has a skewfield of fractions $\mathbb{D}_1(K)$ obtained by localizing $\mathbb{A}_1(K)$ at $S = \mathbb{A}_1(K) - \{0\}$.

   In case the set of all regular elements of A form an Ore set S then we call $S^{-1}A$ the **total fraction ring** of A and we write $S^{-1}A = Q_{\mathbf{cl}}(A)$.

3. Let A be a domain and a K-algebra over a field K. For $x \in A$ we have a K-linear $ad(x) : A \to A, a \mapsto xa - ax$, write $[x,a] = ad(x)(a)$ for $a \in A$. We say that $ad(x)$ is **locally nilpotent** if for every $a \in A$ there is a K-vectorspace W in A, such that $\dim_K W < \infty, a \in W$, and W is stable for $ad(x)$ while $ad(x)|W$ is nilpotent. If $ad(x)$ is locally nilpotent then $S = \{1, x, x^2, \ldots\}$ satisfies the left and right Ore conditions. It suffices to establish that there is an $m \in \mathbb{N}$ such that $ax^m \in xA$ for $a \in A$ (indeed, put $ax^m = xb$ then $bx^n = xc$ for certain n and $c \in A$, thus $ax^{m+n} = x^2c$ etc..., hence for x^d there is an x^N such that $ax^N = x^dz$ for some $z \in A$). Pick $a \in A$. Let V be the finite dimensional K-space generated by the $ad(x)^j(a)$, for $j = 0, 1, \ldots, n$....

   Since $ad(x)|V$ is nilpotent we obtain a finite sequence : $0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$, such that $ad(x)$ is the 0-map on each V_i/V_{i-1}, $i = 1, \ldots, n$. For $v \in V_i$ we have : $vx = xv - [x,v] \in xv + V_{i-1}$, because $ad(x)(v) \in V_{i-1}$ if $v \in V_i$ by definition of the sequence. Hence $V_ix \subset xA + V_{i-1}$. Therefore we obtain : $ax^n + V_nx^n \subset xA + V_{n-1}x^{n-1} \subset \ldots \subset xA + V_1x \subset xA$ (indeed, $xA + V_0 = xA$), where n is the length of the chain in V. This proof can be generalized directly to $\{x_i, i \in J\} \subset A$ with $x_ix_j = x_jx_i$ and $ad(x_i)$ locally nilpotent for all $i \in J$ where then $S = (x_{i_1}^{\alpha_1} \ldots x_{i_p}^{\alpha_p}, p \in \mathbb{N}, \alpha_j \in \mathbb{N}, i_j \in J\}$ is again an Ore set.

   Now we want to localize not only A at S but an arbitrary (left) A-module.

   For a left A-module M and a left Ore set S in A we define the **quotient module with denominator in** S by taking on $S \times M$ the relation N defined by $(s,m) \sim (t,n)$ if and only if there are u, v in A such that $us = vt \in S$ and $um = vn$. Write $s^{-1}m$ for the class of (s, m) with respect to the equivalence (!) relation $\sim$. Let $j_M : M \to S^{-1}M$ be the canonical map $m \mapsto (1, m)/\sim$. It is obvious that $\mathrm{Ker}(j_M) = \{m \in M, sm = 0$ for some $s \in S\} = t_S(M)$; the latter is called the S-**torsion part** of M, it is clearly an A-submodule (follows from the left Ore condition). We define a scalar multiplication by $S^{-1}A$ on $S^{-1}M$ in the way the product of $S^{-1}A$ was defined, this makes $S^{-1}M$ into a left $S^{-1}A$-module. Observe that $S^{-1}M \cong S^{-1}(M/t_S(M))$. Fom $s((s,m)/\sim) = (1,m)/\sim$ it follows that $(s,m)/\sim = s^{-1}((1,m)/\sim)$; hence every $S^{-1}A$-submodule N of $S^{-1}M$ may be obtained as $S^{-1}(j_M^{-1}(N))$ and $N = S^{-1}A(N \cap \overline{M})$ where $\overline{M} = M/t_S(M)$. Observe that $M/j_M^{-1}(N)$ is an S-torsionfree A-module, i.e. $sm \in j_M^{-1}(N)$ implies $j_M(sm) = sj_M(m) \in N$ hence $j_M(m) \in N \cap \overline{M}$ or $m \in j_M^{-1}(N)$.

5.3. Proposition

Let S be an Ore set in A, M a left A-module, then $S^{-1}M \cong S^{-1}A \otimes_A M$.

Proof
Hint : Look at the canonical map $S^{-1}A \otimes_A M \to S^{-1}M$, $((s,a)/\sim) \otimes m \mapsto (s, am)/\sim$; verify that this is a bijection of A-modules but also as $S^{-1}A$-modules. You may replace M by $\overline{M} = M/t_S(M)$ because $S^{-1}A \otimes_A t_S(M) = 0$.

□

5.4. Proposition With notation as before, if M is a (l) Noetherian left A-module then $S^{-1}M$ is a (l) Noetherian left $S^{-1}A$-module. If A is a (l) Noetherian ring then $S^{-1}A$ is a (l) Noetherian ring.

Proof Let M be a left Noetherian A-module (the proof for right Noetherian is the same up to interchanging left and right).

Look at an ascending chain of left $S^{-1}A$-modules in $S^{-1}M$: $N_0 \subset \ldots \subset N_i \subset N_{i+1} \subset \ldots \subset S^{-1}M$. Since $\overline{M}$ is also a left Noetherian A-module, the chain in $\overline{M}$:

$$\overline{M} \cap N_0 \subset \ldots \subset \overline{M} \cap N_i \subset \qquad \subset \overline{M}$$

is stationary. Say $\overline{M} \cap N_i = \overline{M} \cap N_{i+1} = \ldots$. Since $N_i = S^{-1}A(\overline{M} \cap N_i)$ it then follows that $N_i = N_{i+1} = \ldots$, hence $S^{-1}M$ is a left Noetherian $S^{-1}A$-module. In particular for $M = A$ it follows that $S^{-1}A$ is a left Noetherian ring if A is left Noetherian. □

5.5. Lemma

Assume A has a (left) ring of fractions Q with denominators S, such that $Q = S^{-1}A$ is left Noetherian. For every ideal I of A the left ideal QI is an ideal of Q.

Proof We have to establish that $QIQ \subset QI$. Since I is an ideal of A it will suffice to establish that $QIs^{-1} \subset QI$ for every $s \in S$. From $Is \subset I$ it follows that $I \subset Is^{-1}$ and we obtain an ascending chain of left ideals of Q :

$$QI \subset QIs^{-1} \subset QIs^{-2} \subset \ldots \subset QIs^{-n} \subset \ldots$$

Since $S^{-1}A = Q$ is assumed to be left Noetherian, $QIs^{-n} = QIs^{-n-1} = \ldots$, for certain $n \in \mathbb{N}$. Thus $QI = QIs^{-1}$ in Q and hence QI is an ideal of Q. $\square$

We want to prove that a semiprime Noetherian ring is an order in a semisimple Artinian ring; this is the celebrated Goldie theorem. We start with a lemma. A submodule N of M is **essential in** M if for every nonzero submodule L of M we have $N \cap L \neq 0$.

5.6. Lemma

Let A be a semiprime Noetherian (left) ring. For $a \in A$ let $l(A) = \{b \in A, ba = 0\}$.

1. If $l(a)$ is essential in A then $a = 0$.
2. If $l(a) = 0$ then Aa is essential and a is regular.
3. Every essential left ideal L of A contains a regular element.

Proof

1. Suppose $a \neq 0$; we may assume without loss of generality that $l(a)$ is maximal in the set of left annihilator left ideals (A is left Noetherian !).

   Consider the right annihilator N of $l(a)$. We will show that for $z \in N, z^2 = 0$, hence $N^2 = 0$ or $N \subset \cap\{P,$ prime ideal of $A\}$ hence as A is semiprime $N = 0$ but that contradicts $l(a)a = 0$ i.e. $a \in N$ and we assumed $a \neq 0$. Look at $b \in N$ such that $b^2 \neq 0$, then : $l(a) \subset l(b) \subset l(b^2)$. By the choice of a, it follows that $l(a) = l(b) = l(b^2)$. Since $l(a) = l(b)$ is essential in A we have $l(b) \cap Ab \neq 0$, thus there is an $x \in A$ such that $xb \neq 0$ and $xb.b = xb^2 = 0$ but that will contradict $l(b) = l(b^2)$.

2. Assume L is a left ideal of A such that $L \cap Aa = 0$. Then $L \cap La = 0$ and $L \oplus La$ is a direct sum. Since $l(a) = 0$ we also have $La \oplus La^2 = La + La^2, \ldots,$ etc. .... Hence all sums $L \oplus La \oplus \ldots \oplus La^n$, $n \in \mathbb{N}$, are direct and thus if $L \neq 0$ the foregoing contradicts the Noetherian hypothesis. Consequently $L = 0$ and Aa is essential in A. If $ax = 0$ then $Aa \subset l(x)$ and then $l(x)$ is essential in A too, from 1. it follows then that $x = 0$, thus a is regular because $l(a) = 0$ and $r(a) = \{x \in A, ax = 0\} = 0$.

3. Since $L \neq 0$ it contains a non-nilpotent element $a \in L$. Up to replacing a by a^n we may assume that $l(a) = l(a^2)$ (using the *acc* on the chain $l(a) \subset l(a^2) \subset \ldots \subset l(a^m) \subset \ldots$). From $xa^2 = 0$ then $xa = 0$ follows, thus $Aa \cap l(a) = 0$. If $l(a) = 0$ then from 2. we derive that Aa is essential in A hence L contains the regular a. So assume now $l(a) \neq 0$, then $L_1 = L \cap l(a) \neq 0$. Repeat the argument at the beginning of the proof of 3. for L_1, so we find an $a_1 \in L_1$ such that $l(a_1) = l(a_1^2) \neq A$ with $Aa_1 \cap l(a_1) = 0$. From $Aa \cap l(a) = 0$, $a_1 \in l(a)$ and $Aa_1 \cap l(a_1) = 0$ we then obtain a direct sum in $A :$ $Aa \oplus Aa_1 \oplus L_2$ where $L_2 =$

$L_1 \cap l(a_1)$. If $L_2 \neq 0$ we continue until we reach a direct sum in $A: Aa \oplus Aa_1 \oplus \ldots \oplus Aa_n \subset A$ with $a_{i+1} \neq 0$ in L_{i+1}, $L_{i+1} = L \cap l(a) \cap l(a_1) \cap \ldots \cap l(a_i), 1 \leq i \leq n-1$.

Since A is left Noetherian there exists an n such that $L_{n+1} = L_n \cap l(a_n) = 0$. The element $t = a_1 + \ldots + a_n + a \in L$ is regular in A because if $xt = xa + xa_1 + \ldots + xa_n = 0$ then the directness of $Aa \oplus Aa_1 \oplus \ldots \oplus Aa_n$ entails that $xa = \ldots = xa_n = 0$ and thus $l(t) = l(a) \cap \ldots \cap l(a_n)$. From $0 = L_{n+1} = L \cap l(t)$ then it follows that $l(t) = 0$ what implies that t is regular in view of 2. established before.

□

5.7. Theorem. Goldie's theorem for Noetherian rings

For a (l) Noetherian ring A the following statements are equivalent :

1. The total quotient ring $Q_{cl}(A)$ exists and it is a semisimple (resp. simple) Artinian ring.
2. The ring A is semiprime (resp. prime).

Proof

- **1. $\Longrightarrow$ 2.** Let $S = A_{\text{reg}}$ be the set of regular elements of A and $Q_{cl} = S^{-1}A$ is semisimple Artinian. Since the nilradical N of A is nilpotent, the right annihilator I of N is an essential left ideal of A; indeed for $x \neq 0$ in A there is an $n \in \mathbb{N}$ such that $N^n x \neq 0$ and $N^{n+1}x = 0$, thus $0 \neq N^n x \subset I \cap Ax$. Since A is essential in $S^{-1}A$, I is essential in $S^{-1}A$ and thus $S^{-1}I$ is essential in $S^{-1}A$. However since $S^{-1}A$ is semisimple Artinian the left essential ideals are trivial, hence $S^{-1}I = S^{-1}A$. From $1 \in S^{-1}I$ it follows that $1.s \in I$ for some $s \in S$, but the latter s is regular and therefore $N = 0$ or A is semiprime. If $S^{-1}A$ is semisimple then look at ideals J_1 and J_2 in A such that $J_1 J_2 = 0$. Since $J^{-1}A$ is Noetherian, $S^{-1}J_1$, and $S^{-1}J_2$ are ideals of $S^{-1}A$, thus $(S^{-1}J_1)(S^{-1}J_2) = S^{-1}(J_1 J_2) = 0$ yields $S^{-1}J$ or $S^{-1}J_2$ equals zero, consequently J_1 or J_2 is zero and therefore A is a prime ring.

- **2. $\Longrightarrow$ 1.** Start from the assumption that A is a semiprime (l) Noetherian ring. We have to establish that $S = A_{\text{reg}}$ satisfies the Ore condition. Take $s \in S$ and $a \in A$. We show that $L = \{x \in A, xa \in As\}$ is essential in A.

  Pick $y \neq 0$ in A; if $ya = 0$ then $Ay \subset L$. If $ya \neq 0$ then $Aya \cap As \neq 0$ because As is essential (2. in foregoing Lemma). Take $xya \neq 0$ in As, then $0 \neq xy \in L \cap Ay$. Therefore L is essential in A and from 3. in the foregoing lemma it follows that $S \cap L \neq \emptyset$, thus $Sa \cap As \neq \emptyset$ and the Ore condition has been established, hence $Q = S^{-1}A$ exists. Take a left ideal L or Q. If we can prove that L has a complement, i.e. there is a left ideal L' such that $L \oplus L' = Q$, then Q will be semisimple Artinian (verify this, it is a well-known characterization of semisimple Artinian rings). Using Zorn's lemma, we obtain a left ideal L' maximal such that $L \cap L' = 0$. Then $L \oplus L'$ is essential in Q because if $U \cap (L + L') = 0$ then also $L \cap (L' + U) = 0$, hence $U \subset L'$ or $U = 0$. Then $(L + L') \cap A$ is essential in A, this is clear because we have $L + L' = S^{-1}A((L + L') \cap A))$ and if $U \cap ((L + L') \cap A) = 0$ then $S^{-1}AU \cup (L + L') = 0$ hence $S^{-1}AU = 0$ and $U = 0$.

Part 3. of the foregoing lemma yields that there exists $t \in S \cap ((L + L') \cap A)$ and t is invertible on Q, therefore $L + L' = Q$. Now suppose that moreover A is a prime ring and let J_1, J_2 be ideals in Q such that $J_1 \cap J_2 = 0$. Put $I_1 = J_1 \cap A, I_2 = J_2 \cap A$ then $I_1 \cap I_2 = 0$ and $I_1, I_2 \neq 0$ because A is essential Q. We arrive at $I_1 = 0$ or $I_2 = 0$ and thus $J_1 = 0$ or $J_2 = 0$. Thus we may conclude that Q is prime but since it is also Artinian it must be simple (prime ideals are maximal).

□

5.8. Lemma

Assume that $Q_{\mathrm{cl}}(A)$ exists and it is a simple Artinian ring. If S is a multiplicative set of A satisfying the left Ore conditions then S consists of regular elements of A (hence $S^{-1}A \subset Q_{\mathrm{cl}}(A)$).

Proof We have $Q_{\mathrm{cl}}(A) = M_n(\Delta)$ thus a left zerodivisor is also a right zerodivisor and conversely. Suppose S contains a non-regular element $s \in S$ with $l_Q(s) \neq 0, r_Q(s) \neq 0$.

Choose s such that $r_Q(s)$ is maximal between right annihilators in Q. Up to replacing s by s^n (this does not change the $r_Q(s)$ in view of the maximality) we may assume that $l_Q(s) = l_Q(s^2)$.

Let $a \neq 0$ in Q be such that $as = 0$; up to multiplying by some regular element of A we may assume that $a \in A$. As Q is simple we have that A is prime. From $s^2 \in S$ it follows that $s^2 \neq 0$ and as A is prime there is an $x \in A$ such that $s^2xa \neq 0$. Apply the (left) Ore condition on s and $a' = s^2xa$, this yields $t \in S$ and $b \in A$ such that $ta' = bs$. From $as = 0$ we obtain : $ta's = 0 = bs^2$. Thus $l_A(s^2) \subset l_Q(s^2) = l_Q(s)$, and it follows that $bs = 0 = ta' = (ts)sxa$. Consequently $r_Q(ts) \supsetneqq r_Q(s)$ because $sxa \subset r_Q(ts)$ but $sxa \notin r_Q(s)$ (since $s^2xa \neq 0$). But the latter contradicts the maximality of $r_Q(s)$ □

5.9. Proposition

Let A be a (l) Noetherian ring and S a left Ore set. The map $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A), P \mapsto R \cap A$, is injective and has the image $\{Q \in \mathrm{Spec}(A), Q \cap S = \emptyset\}$. The inverse map is $Q \mapsto S^{-1}Q$. The total ring of fractions of $A|P \cap A$ is isomorphic to the one of $S^{-1}A|P$, for $P \in \mathrm{Spec}S^{-1}A$.

Proof Since $S^{-1}A$ is a (l)Noetherian ring, ideals of A extend on the left to ideals of $S^{-1}A$. It is easy now to derive from this the correspondence between prime ideals as claimed. If $p \in \mathrm{Spec}(A)$ is such that $p \cap S = \phi$ then $S^{-1}Ap = P$ is a prime ideal of $S^{-1}A$ and $\overline{S}$, the image of S in A/p (a Noetherian prime ring) consists of regular elements of A/p. We have $\overline{S}^{-1}(A/p) = S^{-1}A/P$ hence the total rings of fractions of these are also equal, they are obtained by localization at $T = (A/p)_{\mathrm{reg}}$!

5.10. Corollary

Let I be an ideal in a (left) Noetherian ring A and $\varphi : A \to Q_{\mathrm{cl}}(A/\sqrt{I})$ the canonical ring homomorphism (where $\sqrt{I}$ stands for the prime radical of I).

The map $P \to \varphi^{-1}(P)$ is a bijection between the set of minimal prime ideals of A containing I and the prime ideals of $Q_{\mathrm{cl}}(A/\sqrt{I})$.

Proof From the foregoing since $A/\sqrt{I}$ is a semiprime Noetherian ring.

5.11. Remark

From the results (e.g. Lemma 5.6.) and the "right"-version of them obtained by replacing $l(a)$ by $r(a)$ it follows also that in a semiprime Noetherian ring A an $x \in A$ is not a right zerodivisor if and only if it is a non left zerodivisor.

5.12. Corollary

Since $\mathbb{A}_n(K)$ is a Noetherian domain, $S = \mathbb{A}_n(K)_{\mathrm{reg}}$ satisfies the Ore conditions (left and right) and $S^{-1}\mathbb{A}_n(K) = \mathbb{D}_n(K)$ is a simple Artinian domain hence a skewfield.

The following are Ore sets in $\mathbb{A}_1(K)$ (check this !).

a. $S(x) = \{1, x, x^2, \ldots, x^n, \ldots\}$

b. $S(y) = \{1, y, y^2, \ldots, y^n, \ldots\}$

c. $S(xy) = \{1, xy, (xy)^2, \ldots, (xy)^n, \ldots\}$

It is not difficult to calculate the localizations of $\mathbb{A}_1(K)$ at these Ore sets, we leave this as an exercise here.

Chapter 6

Gelfand-Kirillov Dimension and Filtered Rings

6.1 Gelfand-Kirillov Dimension

Consider $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, f(n) \geq 0$ and $f(n+1) - f(n) \geq 0\}$. For $f, g \in \mathcal{F}$ we write $f \leq g$ if there is an $m \in \mathbb{N}$ such that $f(n) \leq g(mn)$ for almost all $n \in \mathbb{N}$. We put $f \sim g$ if $f \leq g$ and $g \leq f$; it is easy to verify that $\sim$ defines an equivalence relations on $\mathcal{F}$. We write $G(f)$ for the class of f in $\mathcal{F}/\sim$ and call it the **growth** of f. The relation $\leq$ on $\mathcal{F}$ induces a partial order on $G(\mathcal{F}) = \mathcal{F}/\sim$; we write $G(f) < G(g)$ if $G(f) \leq G(g)$ and $G(f) \neq G(g)$.

For $\gamma \geq 0$ we denote the growth of $n \mapsto n^\gamma$ by P_γ, the growth of $n \mapsto \exp(n^\gamma)$ is denoted by ε_γ for $\gamma > 0$. It is obvious that $\varepsilon_\varepsilon < \varepsilon_\delta$ if and only if $\varepsilon < \delta$ (indeed, from $\varepsilon \geq \delta$ it follows that $n^\varepsilon < (mn)^\delta$ for almost all $n \in \mathbb{N}$ implies $n^{\varepsilon-\delta} < m^\delta$ for almost all $m \in \mathbb{N}$, a contradiction).

If f and g are polynomial functions in $\mathcal{F}$, that is there are polynomials $F, G \in \mathbb{R}[X]$ such that $f(n) = F(n), g(n) = G(n)$, then $G(f) = G(g)$ if and only if f and g have the same "degree" i.e. $\deg_X F = \deg_X G$! Indeed if $\deg_X F \leq \deg_X G$ then from $G(g) \leq G(f)$ it follows that $g(n) \leq f(mn)$ for some $m \in \mathbb{N}$ and for almost all $n \in \mathbb{N}$ but then $\deg_X G \leq \deg_X F$ because if not, $g(n) > f(mn)$ for large n, thus $\deg_X F = \deg_X G$. For the function $f : n \mapsto \log n$ in $\mathcal{F}$ we have : $P_0 < G(f) < P_\varepsilon$, for all $\varepsilon > 0$ (because : $1 < \log n < n^\varepsilon$ for almost all $n \in \mathbb{N}$).

Now look at a K-algebra generated by $\{a_1, \ldots, a_m\}$ and let V be a K-vectorspace in A. We write V^n for the subspace in A generated by the monomials $a_{i_1} \cdot \ldots \cdot a_{i_n}$, $i_j \in \{1, \ldots, m\}$, and we put $V^0 = K$. Notation : $A_n = \sum_{j=0}^n V^j$. If A is a finitely generated K-algebra then there exists a finite dimensional K-space V in A such that $A = \cup_n A_n$. Such K-vectorspaces will be called **generating vectorspaces for** A.

We have $K = A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots \subset A$ and for $i, j \in \mathbb{N}$ we have $A_i A_j \subset A_{i+j}$, hence the foregoing chain yields a positive filtration on A. The function $d_V : n \mapsto d_V(n)$, $d_V(n) = \dim_K A_n$ is a positive ascending function, hence d_V is in $\mathcal{F}$. Up to replacing V by $K + V$ we may assume that $1 \in V$, then $V^n = \sum_{i=0}^n V^i$.

6.1.1 Lemma

Let A be a finitely generated K-algebra with generator verctorspaces V and W, then : $G(d_V) = G(d_W)$.

Proof Since $A = \cup_n(\sum_{i=0}^n V^i) = \cup_n(\sum_{i=0}^n W^i)$ there are s, t such that $W \subset \sum_{i=0}^s V^i, V \subset \sum_{j=0}^t W^j$. Therefore we obtain : $d_W(n) \leq d_V(sn)$ and $d_V(n) \leq d_W(tn)$, or $G(d_V) = G(d_W)$. □

6.1.2 Definition

Let A be a finitely generated K-algebra A. The **growth function of** A, say $G(A)$, is the $G(d_V)$, where $d_V(n) = \dim_K(\sum_{i=0}^n V^i)$, for some generating vector space V for A.

6.1.3 Remark

If A is finitely generated over K but not finite dimensional then : $P_1 \leq G(A) \leq \varepsilon_1$.

Proof Let V be a generating vectorspace for A with $1 \in V$, then : $\dim_K V^n \leq (\dim_K V)^n$, thus $G(A) \leq \varepsilon_1$. On the other hand we have that $V^n \subsetneqq V^{n+1}$, otherwise $A = V^n$ would make A finite dimensional over K, thus $d_V(n) \geq n$ and $G(A) \geq P_1$. □

Algebras such that $G(A) = P_d$ for some $d \in \mathbb{N}$, are called algebras of **polynomial growth**.

For a real number a we put $\log_n a = \log\, a/\log\, n$; we use the notation $\overline{\lim}$ for the upper-limit.

6.1.4 Lemma

For f and g in $\mathcal{F}$ we have :

i)

$$\begin{aligned} \overline{\lim}\, \log_n f(n) &= \inf\{p \in \mathbb{R}, f(n) \leq n^p \text{ for almost all } n \in \mathbb{N}\} \\ &= \inf\{p \in \mathbb{R}, G(f) \leq P_p\} \end{aligned}$$

ii) If $G(f) = G(g)$ then $\overline{\lim}\, \log_n f(n) = \overline{\lim}\, \log_n(g(n))$.

Proof

i) Put $r = \overline{\lim}\, \log_n f(n)$, $s = \inf\{p \in \mathbb{R}, f(n) \leq n^p$, almost all $n\}$ $t = \inf\{p \in \mathbb{R}, G(f) \leq P_p\}$. If one of these numbers is infinite then so are the others. Obviously, $t \leq s$. Pick $\varepsilon > 0$, then $\log_n f(n) \leq r + 1$ for n large enough, thus $s \leq r$ because $s \leq r + \varepsilon$ for every $\varepsilon > 0$. So we arrive at $t \leq s \leq r$. Suppose $r > t$ and put $\varepsilon = (r - t)/3$; then we have : $G(f) \leq P_{t+\varepsilon}$, or $f(n) \leq (mn)^{t+\varepsilon}$ for some $m \in \mathbb{N}$ and n large enough. Choose n large enough such that $m^{t+\varepsilon} \leq n^\varepsilon$, then we find that $f(n) \leq n^{t+2\varepsilon}$ for n large enough. But $t + 2\varepsilon < r$ would then contradict $r = \overline{\lim} \log_n f(n)$, hence $r = t$ follows.

ii) Easy from i) since $\inf\{p \in \mathbb{R}, G(f) \leq P_p\} = \inf\{p \in \mathbb{R}, G(g) \leq P_p\}$.

$\square$

6.1.5 Definition

Let A be a K-algebra.
The Gefand-Kirillov dimension (for short : GK-dim) of A is : $GK\dim A = \sup_V\{\overline{\lim}\log_n(d_V(n))\} \in \mathbb{R}_+ \cup \{\infty\}$ where V varies over the finite dimensional K-vectorspaces in A.

6.1.6 Lemma

Let B be a finitely generated K-algebra and V a generating vector space, then : $GK\dim B = \overline{\lim}\log_n d_V(n)$. For a K-algebra A we obtain that $GK\dim(A)$ equals $\sup_B\{GK\dim B, B$ a finitely generated K-algebra in $A\}$.

Proof The first statement follows from Lemma 6.1.1. and Lemma 6.1.4. The second statement follows from foregoing. $\square$

6.1.7 Proposition

1. Let A be a K-algebra, B either a K-subalgebra or a homomorphism of A, then $GK\dim B \leq GK\dim A$.

2. If A_1 and A_2 are K-algebras then we have :
   $GK\dim(A_1 \oplus A_2) = \text{Max}\{GK\dim A_1, GK\dim A_2\}$.

3. If $I_1, \ldots, I_d$ are ideals of the K-algebra A, then we obtain :
$$GK\dim(A/I_1 \cap \ldots \cap I_d) = \text{Max}\{GK\dim A/I_j, i \leq j \leq d\}$$

Proof

1. For $B \subset A$ the statement is clear. Now suppose $\varphi : A \to B$, $B = \text{Im}\varphi$, is a ring epimorphism. Let $W \subset B$ be a K-vectorspace with basis $\{\varphi(a_1), \ldots, \varphi(a_m)\}$. Put $V = \sum_{j=1}^m Ka_j$. Then we have $\dim_K V^n \geq \dim W^n$ for all n; the statement follows from the definition of GK-dimension.

2. From 1. it follows that $\mu = \max\{GK\dim A_1, GK\dim A_2\} \leq GK\dim(A_1 \oplus A_2)$. If $\mu = \infty$ the equality holds, therefore assume $\mu < \infty$ and let $W \subset A_1 \oplus A_2$ be a finite dimensional K-vectorspace. Put $U = p_1(W), V = p_2(W)$ where $p_i : A_1 \oplus A_2 \to A_i$ are the canonical projections $i = 1, 2$. For all $n \in \mathbb{N}$ we have $W^n \subset U^n \oplus V^n$. Take $\varepsilon > 0$, Lemma 6.1.4. yields $d_V(n) \leq n^{\mu+\frac{\varepsilon}{2}}, d_U(n) \leq n^{\mu+\frac{\varepsilon}{2}}$, for n large enough. Since, for large enough n we obtain $n^{\varepsilon/2} > 2$ we also find that :
$$d_W(n) \leq d_U(n) + d_V(n) \leq 2n^{\mu+\varepsilon/2} \leq n^{\mu+\varepsilon}$$
   for n large enough. Hence $\overline{\lim}\log_n d_W(n) \leq \mu$, or $GK\dim(A_1 \oplus A_2) \leq \mu$.

3. Follows from 1. and 2. because we have a K-algebra embedding $A/I_1 \cap \ldots \cap I_d \hookrightarrow \oplus_{j=1}^{d} A/I_j$. $\square$

For a K-derivation of a K-algebra A, say δ, we consider the Ore extension $A[X,\delta] = \{\sum_{i=0}^{n} a_i X^i, a_i \in A\}$ with multiplication rule $[X,a] = Xa - aX = \delta(a)$ for all $a \in A$.

6.1.8 Proposition

1. For a K-derivation δ of a K-algebra A, $GK\dim A[X,\delta] \geq GK\dim A + 1$

2. If every finite dimensional K-subspace of A is contained in a finite dimensional δ-invariant K-subspace of A, then $GK\dim A[X,\delta] = 1 + GK\dim A$.

Proof

1. Let B be a finitely generated K-subalgebra in A and V a generating vectorspace for B, with $1 \in V$. Then $W = V + KX$ is a finite dimensinonal K-vectorspace in $A[X,\delta]$ such that : $\sum_{i=0}^{n} V^n X^i \subset (V + KX)^{2n} = W^{2n}$. Thus we obtain $(n+1)d_V(n) \leq d_W(2n)$, thus also :
$$GK\dim A[X,\delta] \geq \overline{\lim}\ \log_n((n+1)d_V(n)) = 1 + GK\dim B$$
By taking sup over all such B we arrive at statement 1.

2. Let B be a finitely generated K-subalgebra in $A[X,\delta]$. The K-space generated in A by the coefficients of a finite number of generators for B is by assumption contained in a finite dimensional δ-invariant subspace of A. The latter generates a finitely generated subalgebra A' of A. It suffices to establish that $GK\dim B \leq GK\dim A' + 1$, thereby reducing the problem to the case where A is finitely generated (passing from A to A'). Let V be a generating finite dimensional subspace in A, with $1 \in V$, then $W = V + KX$ is a finite dimensional generating subspace for $A[X,\delta]$. We have $\delta(V) \subset V^m$ for some $m > 0$, thus $\delta(V^k) \subset V^{m+k}$ for all $k \in \mathbb{N}$. We will now establish the following :
$$(*) \qquad W^n \subset V^{mn} + V^{mn}X + \ldots + V^{mn}X^n = \sum_{i=0}^{n} V^{mn}X^i$$
From (*) it will follow that $d_W(n) \leq (n+1)d_V(mn)$, or $GK\dim A[X,\delta] = \overline{\lim}\ \log_n d_W(n) \leq \overline{\lim}\ \log_n(n+1)d_V(mn) = 1 + GK\dim A$. The proof of (*) is by induction on n. First $(*)$ holds obviously for $n = 0$. Assume $(*)$ holds for some $n \in \mathbb{N}$, then :
$$\begin{aligned} VW^n &\subset \sum_{i=0}^{n} V^{mn+1}X^i \subset \sum_{i=0}^{n} V^{m(n+1)}X^i \\ XW^n &\subset \sum_{i=0}^{n} XV^{mn}X^i \subset \sum_{i=0}^{n} V^{mn}X^{i+1} + \sum_{i=0}^{n} \delta(V^{mn})X^i i \\ &\subset \sum_{i=0}^{n+1} V^{m(n+1)}X^i \end{aligned}$$

the latter inclusion following from $\delta(V^{mn}) \subset \sum_{j=0}^{mn-1} V^j\delta(V)V^{mn-j-1}$ with $\delta(V) \subset V^m$, yielding indeed that

$$\delta(V^{mn}) \subset \sum_{i=0}^{n+1} V^{m(n+1)}X^i$$

Both foregoing inclusions may be combimed into

$$W^{n+1} = (V + KX)W^n \subset \sum_{i=0}^{k+1} V^{m(n+1)}X^i$$

this establishes (*) for $n+1$. □

6.1.9 Corollary

Let A be a K-algebra and $X_1, \ldots, X_n$ commuting variables, then :

$$GK\dim A[X_1, \ldots, X_n] = GK\dim A + n$$

Proof We write $A[X_1, \ldots, X_n] = A[X_1, \ldots, X_{n-1}][X_n, 0]$, i.e. an Ore extension with $\delta = 0$. Applying the foregoing proposition n times, the statement follows. □

We have seen that $\mathbb{A}_n(K) = K[x_1, \ldots, x_n][y_1, \ldots, y_n, \partial_1, \ldots, \partial_n]$. In particular $\mathbb{A}_n(K) = \mathbb{A}_{n-1}(K)[X_n, \frac{\partial}{\partial X_n}]$.

6.1.10 Proposition

$GK\dim\mathbb{A}(K) = 2n$.

Proof

$$\begin{aligned} GK\dim\mathbb{A}_n(K) &= GK\dim\ \mathbb{A}_{n-1}(K)[x_n] + 1 \\ &= GK\dim\ \mathbb{A}_{n-1}(K) + 2 \end{aligned}$$

Repeating the argument for $\mathbb{A}_{n-1}(K)$, and so on, we arrive at $GK\dim\mathbb{A}_n(K) = 2n$. □

6.1.11 Particular case : $GK\dim\mathbb{A}_1(K) = 2$

Let A be a Noetherian K-algebra, A_{reg} is the set of left and right regular elements of A. Obviously A_{reg} is multiplicatively closed and $1 \in A_{\text{reg}}$.

6.1.12 Proposition

Let A be a Noetherian K-algebra.

1. If I is an ideal of A such that $I \cap A_{\text{reg}} \neq \emptyset$ then : $GK\text{dim}A/I \leq GK\text{dim}A - 1$.
2. Take $P \in \text{Spec}(A)$ and suppose $htP = n$, $htP = \sup_m \{P = P_0 \supsetneqq P_1 \supsetneqq \ldots \supsetneqq P_m, P_j \in \text{Spec}(A)\}$, then we have : $GK\text{dim}A/P \leq GK\text{dim}A - n$.

Proof

1. Take $c \in A_{\text{reg}} \cap I$ and put $\overline{A} = A/I$, let $\overline{V} \subset \overline{A}$ be a finite dimensional vector space with $1 \in \overline{V}$. Let $V \subset A$ be the vector space generated by $1, c$ and $a_1, \ldots, a_d$, where $\{\overline{a}_1, \ldots, \overline{a}_d\}$ is a K-basis for $\overline{V}$. For all n let S_n be a supplement of $I \cap V^n$ in V^n, hence $S_n \cong \overline{V}^n = \overline{V}^n$. Therefore $S_n \cap cA = 0$ and $\sum_{i=0}^n c^i S_n$ is a direct sum in V^{2n}; thus we obtain : $\dim_K V^{2n} \geq n\dim_K S_n = n\dim_K \overline{V}^n$. Therefore :
$$GK\text{dim}A/I + 1 = \sup_{\overline{V}} \overline{\lim} \log_n d_{\overline{V}}(n) + 1 =$$
$$\sup_{\overline{V}}\{\overline{\lim} \log_n(nd_{\overline{V}}(n))\} \leq \sup_{\overline{V}}\{\overline{\lim \log_n} d_V(2n)\} \leq GK\text{dim}A$$
2. Suppose $P = P_0 \supsetneqq \ldots \supsetneqq P_n$ is a maximal chain of prime ideals of A. For every i, P_i/P_{i+1} is an essential ideal in A/P_{i+1} (if L_{i+1} is a right ideal in A/P_{i+1} then $L_{i+1}\overline{P}_i \subset L_{i+1} \cap \overline{P}_i$ and $L_{i+1}\overline{P}_i \neq 0$ since A/P_{i+1} is a prime ring), thus P_i/P_{i+1} contains an element of A_{reg}. From 1. it follows that $GK\text{dim}A/P_i \leq GK\text{dim}A/P_{i+1} - 1$, hence by recurrence, $GK\text{dim}A/P \leq \text{dim}A - m$. □

6.1.13 Lemma

Let S be a central multiplicative set of the K-algebra A consisting of regular elements of A. Then S is an Ore set (left and right) of A and we have : $GK\text{dim}S^{-1}A = GK\text{dim}A$.

Proof As S is central in A it is obviously a left and right Ore set. Let $W \subset S^{-1}A$ be a finite dimensional K-vector space, then there is a common denominator for a finite K-basis, say $s \in S$, such that $sW \subset A$. Put $V = sW + Ks + K$. Then $\dim_K V < \infty$ and $W^n \subset s^{-n}V^n$ for all $n \in \mathbb{N}$ (s is central in A !). Hence $d_W(n) \leq d_V(n)$ and $GK\text{dim}S^{-1}A \leq GK\text{dim}A$. Since S consists of regular elements, $A \subset S^{-1}A$, and thus $GK\text{dim}A \leq GK\text{dim}S^{-1}A$ leading to the desired equality. □

6.1.14 Lemma

Let B be a subalgebra of the K-algebra A. Let A be commutative and finitely generated as a B-module, then we obtain : $GK\text{dim}A = GK\text{dim}B$.

Proof Clearly $GK\text{dim}B \leq GK\text{dim}A$. Write $A = \sum_{i=1}^d Ba_i$ and let V be a finite dimensional subspace of A generated by $\{v_1, \ldots, v_n\}$ containing $\{a_1, \ldots, a_d\}$. We have : $v_i = \sum_k b_{ik}a_k$ and $V_iV_j = \sum_h b_{ijh}a_h$, for $1 \leq i, j \leq d$, with $b_{ik}, b_{ijh} \in B$. Put $W = K.1 + \sum_{i,k} Kb_{ik} + \sum_{i,j,h} Kb_{ijh}$. By recurrence we have $V^n \subset W^{2n-1}a_1 + \ldots + W^{2n.1}a_d$, for all n. Thus $d_V(n) \leq dd_W(2n-1)$ and $GK\text{dim}A \leq GK\text{dim}B$ follows. □

6.1.15 Proposition

1. Let K/k be a field extension, then $GK\dim K = td_k(K)$, the transcendence degree of K over k.

2. Let A be a commutative k-algebra which is finitely generated. Then $GK\dim A = K\dim A$, the latter being the Krull dimension, $K\dim A = \sup_{P\in\mathrm{Spec}(A)}\{ht\ P\}$. In case A is a domain with field of fractions $K = Q_{\mathrm{cl}}(A)$, then $GK\dim A = td_k(K)$.

Proof

1. K is an algebraic extension of a purely transcendented $F = k(X_1, \ldots, X_d)$. Now $GK\dim F = d = td_k(K)$. If A is a finitely generated k-algebra in K then A is algebraic over some finitely generated subfield F in K, hence it is contained in some finite dimensional extension of F and thus $GK\dim A = GK\dim F$. Since F is finitely generated we obtain $GK\dim F = tr_k F \leq td_k K$. Hence $GK\dim K = td_k K$.

2. Noether's normalization lemma yields that A s a finitely generated module over $k[X_1, \ldots, X_r]$ where $r = K\dim A$.

   Thus $GK\dim A = GK\dim k[X_1, \ldots, X_r] = r$. If A is a domain then the statement follows from Lemma 6.1.13. □

6.1.16 Proposition

Let A be a K-algebra and $\{x_i, i \in J\}$ a family regular elements of A such that $x_i x_j = x_j x_i$ and $\mathrm{ad}x_i : a \mapsto [x_i, a]$ is locally nilpotent for all $i \in J$. Let δ be the multiplicative set generated by the $x_i, i \in J$, then : Let δ be the multiplicative set generated by the $x_i, i \in J$, then : $GK\dim S^{-1}A = GK\dim A$.

Proof By example 5.2.3., S is an Ore set in A, so $S^{-1}A$ exists. The proof is now a technical modification of 6.1.13. □

We will now extend the definition of $GK\dim$ to A-modules. This leads to a useful dimension that can be applied to the theory of $\mathcal{D}$-modules, modules over rings of differential operators or more general filtered rings like the class of "almost commutative" rings generalizing the Weyl algebras. The dimension theory is also useful in the study of modules over Weyl algebras, in particular with respect to the so-called holonomic modules and holonomic simple modules. Let A be a finitely generated K-lagebra and M a finitely generated A-module generated by a finite dimensional K-vector space, μ say. For every generating vector space V in A with $1 \in V$ we have : $M = \sum_{n=0}^{\infty} V^n\mu$. Look at the function $n \mapsto d_{V,\mu}(n) = \dim_K V^n\mu$. This growth function does not depend on the choice of μ and V (verify this), we may thus define $G(M) = G(d_{V,\mu})$.

6.1.17 Definition

Let A be a K-algebra and M a left A-module. The **Gelfand-Kirillov dimension of** M, $GK\dim_A M$, is defined by :

$$\sup_{V,\mu}\{\overline{\lim}\log_n(\dim_K V^n\mu)\}$$

where V varies over the finite dimensional K-vector spaces in A and μ varies over the finite dimensional K-vector spaces in M. For right A-modules the right-hand version of the definition may be phrased. We put $GK\dim_A(0) = -\infty$. When $M =_A A$, then $GK\dim_A(_AA) = GK\dim A$ holds.

6.1.18 Proposition

Let A be a K-algebra, M a left A-module.

1. If $M = \oplus_{i=1}^d M_i$ then $GK\dim M = \mathrm{Max}_i\{GK\dim M_i\}$.
2. If we have an exact sequence in A-mod : $0 \to K \to M \to L \to 0$, then $GK\dim M \geq \mathrm{Max}\{GK\dim K, GK\dim L\}$.
3. If I is an ideal of A such that $IM = 0$, then we have : $GK\dim_A M = GK\dim_{A/I} M$, in particular for $l(M) = \{a \in A, aM = 0\}$ we obtain : $GK\dim_A M = GK\dim_{A/l(M)} M$.
4. $GK\dim_A M \leq GK\dim A$, for any $M \in A$-mod.
5. If M is a finitely generated left A-module and $\alpha \in \mathrm{End}_A(M)$ an injective endomorphism then : $GK\dim \frac{M}{\alpha(M)} \leq GK\dim M - 1$.
6. If $M = \sum_{i=1}^d M_i$, then $GK\dim_A M = \mathrm{Max}_i\{GK\dim_A M_i\}$.

Proof

1. Similar to 6.1.7.(2)
2. and 3. Direct consequences of the definition.
4. Follows from $\dim_K V^n\mu \leq \dim_K \mu \dim_K V^n$, first for finitely generated M and after that we may apply $GK\dim A \geq \sup\limits_{N \subset M}\{GK\dim_A N, N \text{ finitely generated in } M\}$.
5. Let $V \subset A$ be a finite dimensional K-vector space and $\overline{\mu}$ a finite dimensional generating K-vector space for $\overline{M} = M/\alpha(M)$. Since $\alpha(M)$ is finitely generated there exists a finite dimensional K-subspace μ of M such that $\overline{\mu}$ is the image of μ under de canonical $M \to \overline{M}$ and such that μ is generating for M. For $n \geq 0$ let C_n be a complement of $\alpha(M) \cap V^n\mu$ in $V^n\mu$, thus $C_n \cong V^n\overline{\mu}$. Since $C_n \cap \alpha(M) = 0$ the sum $\sum_{i=0}^r \alpha^i(C_n)$ is direct for every r (indeed, since α is injective we have $\alpha(C_n \cap \alpha(M)) = \alpha(C_n) \cap \alpha^2(M)$ etc...). On the other hand there is a finite dimensional subspace W in A such that $W \supset V$ and $\alpha(\mu) \subset W\mu$ (because $\dim_K \alpha(N) < \infty$ and $M = A\mu$). Consequently we obtain :
$$\oplus_{j=0}^n \alpha^j(C_n) \subset \oplus_{j=0}^n \alpha^j(V^n\mu) = \oplus_{j=0}^n V^n\alpha^j(\mu) \subset \oplus_{j=0}^n V^n W^j \mu \subset W^{2n}\mu$$
thus also : $\dim_K(W^{2n}\mu) \geq (n+1)\dim_K C_n = (n+1)\dim_K V^n\overline{\mu}$. Then : $GK\dim_A M \geq GK\dim_A\left(\frac{M}{\alpha(M)}\right) + 1$, and the statement follows.
6. Follows from 5. and 2. because $\oplus_i M_i \to M = \sum M_i$ yields $GK\dim_A M \leq GK\dim_A(\oplus_i M_i) = \mathrm{Max}_i\{GK\dim_A M_i\}$. □

For two K-algebras A and B we may look at an (A, B)-bimodule M.

6.1.19 Proposition

Consider an (A, B)-bimodule M, ${}_AM_B$ say, such that the left A-module M is finitely generated.

1. $GK\dim_B M_B = GK\dim(B/r(M)), r(M) = \{b \in B, Mb = 0\}$.
2. $GK\dim_B M_B \leq GK\dim_A M$.
3. If M is finitely generated as a right B-module then $GK\dim_B M_B = GK\dim_A M$.
4. If B is a subalgebra of A and A is finitely generated as a B-module then $GK\dim A = GK\dim B$.

Proof

1. Write $M = \sum_{i=1}^{t} Am_i$.
   Thus $r(M) = \cap_{i=1}^{t} r(m_i)$, $r(m_i) = \{b \in B, m_i b = 0\}$. We obtain a chain of right B-modules : $B/r(M) \hookrightarrow \oplus_{i=1}^{t} B/r(m_i) \hookrightarrow M^{(t)} = M \oplus \ldots \oplus M$. From Proposition 6.1.18 we retain :
   $$GK\dim_B B/r(M) \leq GK\dim_B M_B \leq GK\dim_B B/r(M)$$
2. Let μ and V be finite dimensional K-subspaces of M, resp. A. Since ${}_AM$ is finitely generated there is an $\mathcal{N} \supset \mu$, a finite dimensional K-space such that $A\mathcal{N} = M$. Since $\dim_K \mathcal{N}V < \infty$ and $M = A\mathcal{N} \supset \mathcal{N}V$, there is a finite dimensional K-space $W \subset A$ such that : $W\mathcal{N} \supset \mathcal{N}V$. Thus $\mu V^n \subset \mathcal{N}V^n \subset W^n N$, for all n, and the statement follows. $\square$
3. The left version of 2 now holds too since M is also finitely generated as a B-module, then 3. follows from 2.
4. From 3. we retain $GK\dim A = GK\dim_B A_B$ and also $GK\dim_B A_B \leq GK\dim B$. Since $GK\dim A \geq GK\dim B$, the equality follows. $\square$

6.1.20 Proposition

Let A be a K-algebra, M a left A-module. Consider an A-bimodule N which is finitely generated as a right A-module, then : $GK\dim_A(N \otimes_A M) \leq GK\dim_A M$.

Proof Observe that $N \otimes_A M$ is a left A-module with the structure given by : $a.(n \otimes m) = an \otimes m$, for $a \in A, n \in N$, $m \in M$. Consider finite dimensional K-subspaces $E \subset N \otimes_A M$ and $V \subset N$ with $1 \in V$. There exist finite dimensional K-vector spaces $\mu \subset M$, $\mathcal{N} \subset N$ such that $N = \mathcal{N}A, E \subset \mathcal{N} \otimes_A \mu = \{\Sigma' \eta_i \otimes \mu_i, \eta_i \in \eta, \mu_i \in \mu\}$. Since $\dim_K V\mathcal{N} < \infty$ there is a subspace $W \supset V$ in A such that $\dim_K W < \infty$ and $V\mathcal{N} \subset \mathcal{N}W$, thus $V^n\mathcal{N} \subset \mathcal{N}W^n$ for all n. Similarly : $V^n E \subset V^n(\mathcal{N} \otimes_A \mu) \subset \mathcal{N}W^n \otimes_A \mu = \mathcal{N} \otimes W^n\mu$. Since $\mathcal{N} \otimes_K W^n\mu \longrightarrow\!\!\!\!\rightarrow \mathcal{N} \otimes_A W^n$ as K-vector spaces we obtain : $\dim_K V^n E \leq \dim_K \mathcal{N} \otimes W^n\mu = \dim_K \mathcal{N}.\dim_K W^n\mu$. Finally we thus arrive at :

$$\overline{\lim}\ \log_n \dim_K V^n E \leq \overline{\lim}\ \log_n \dim_K W^n \mu$$

thus : $GK\dim_A(N \otimes_A M) \leq GK\dim_A M$. $\square$

As a special case of the foregoing result we obtain the following.

6.1.21 Corollary

Let A be a Noetherian K-algebras. Apply the foregoing proposition to $N = I$ an ideal of A :

$GK\dim(I \otimes M) \leq GK\dim_A M$
$GK\dim_A(M \otimes I)_A \leq GK\dim_A M$

This property is called the **ideal-invariance** of $GK\dim$.

6.1.22 Definition

Let A be a K-algebra. We say that $GK\dim$ is exact for (left) modules that are finitely generated, if an exact sequence $0 \to L \to M \to N \to 0$ of finitely generated A-modules yields $GK\dim_A M = \mathrm{Max}\{GK\dim_A L, GK\dim_A N\}$.

6.1.23 Proposition

Assume that A is a Noetherian K-algebra such that $GK\dim$ is exact. Then we have : $GK\dim A = GK\dim(A/\sqrt{0}) = GK\dim(A/P)$, for at least one minimal prime ideal P of A, where $\sqrt{0}$ is the nilradical.

Proof Since A is Noetherian $\sqrt{0} = \cap\{P_i, P_i$ minimal prime ideal of $A\}$. The final statement follows thus from Proposition 6.1.7.3. Write $N = \sqrt{0}$, then $N^r = 0$ and $N^{r-1} \neq 0$ for some $r \in \mathbb{N}$. Look at the chain of left A-modules : $A = \mathbb{N}^0 \supsetneqq N \supsetneqq N^2 \supsetneqq \ldots \supsetneqq N^r = 0$, where each N^i/N^{i+1} is an A/N-module. Thus $GK\dim_A N^i/N^{i+1} \leq GK\dim A/N$ (see Proposition 6.1.18(4)). Using repeatedly the exactness of $GK\dim$ for exact sequences $0 \to N^{i-1} \to N^i \to N^i/N^{i+1} \to 0$, we arrive at $GK\dim A \leq GK\dim A/N$, leading to the inequality $GK\dim A \leq GK\dim A/N$ hence to the desired equality because the reverse inequality is obvious. $\square$

6.1.24 Definition

An A-module M is said to be $GK\dim$-**homogeneous** if for every nonzero submodule N of M we have $GK\dim M = GK\dim N$.

6.1.25 Proposition

1. Let A be a prime Noetherian K-algebra, then A is $GK\dim$-homogenous as a left and right A-module.

2. Let A be a K-algebra, M a $GK\dim$-homogeneous A-module and $N \subset M$ a submodule such that $GK\dim M/N < GK\dim M$ then N is essential in M.

Proof

1. Since A is Noetherian we have $LA = \sum_{i=1}^{t} La_i, a_i \in A$, for every nonzero left ideal L of A. The A-linear map $L \oplus \ldots \oplus L \to LA$, $(\lambda_1, \ldots, \lambda_t) \mapsto \sum_{i=1}^{t} \lambda_i a_i$, is surjective, thus $GK\dim_A(L \oplus \ldots \oplus L) = GK\dim_A L \geq GK\dim_A(LA) \geq GL\dim_A L$. From $L \subset LA$ we then derive : $GK\dim_A L = GK\dim_A LA$. Now $LA \neq 0$ is an ideal of A and then it contains a regular element of A, say $c \in LA$. From $Ac \cong A$ and $Ac \subset LA$ it follows that : $GK\dim A \leq GK\dim_A AL \leq GK\dim A$, therefore the equality $GK\dim_A L = GK\dim_A LA = GK\dim A$ follows.

2. If $X \neq 0$ is a sub A-module of M such that $N \cap X = 0$ then $GK\dim M = GK\dim X \leq GK\dim M/N$ because $X \hookrightarrow M/N$. This yields a contradiction thus N must be essential since no X as before can exist. $\square$

6.2 Filtered Rings and $GK\dim$

First we look at a graded agebra. Let A be a $\mathbb{Z}$-graded K-algebra and M a graded left A-module. If for al $i \in \mathbb{Z}, \dim_K A_i < \infty$, resp. $\dim_K M_i < \infty$, then we say that the gradation of A, resp. M, is **finite** (dimensional). Put $A(n) = \oplus_{i=-n}^{n} A_i$, $M(n) = \oplus_{i=-n}^{n} M_i, d_A(n) = \dim_K(A(n))$, $d_M(n) = \dim_K(M(n))$, for $n \geq 0$.

6.2.1 Lemma

1. Consider finite dimensional K-vector spaces $E \subset M, V \subset A$ with $1 \in V$, then : $G(d_{V,E}) \leq G(d_M)$ and $GK\dim_A M \leq \overline{\lim} \log_n d_M(n)$.

2. If A is finitely generated and M is a finitely generated A-module then $G(M) = G(d_M)$ and $GK\dim_A M = \overline{\lim} \log_n d_M(n)$.

Proof Recall that $d_{V,E}$ is defined as $\dim_K(V^n E) = d_{V,E}(n)$ and $G(M) = G(d_{V,E})$, where V is a generating subspace of A and $M = AE$.

1. The second statement follows from the first by talking sup over V and E. Take $p \in \mathbb{N}$ and $V \subset A(p), E \subset M(p)$, then for all $n \geq 1$ we have : $V^n E \subset A(pn)E \subset M(pn+p) \subset M(2pn)$, thus : $d_{V,E}(n) \leq d_M(2pn)$, and this entails the statement 1.

2. Take p large enough such that $V = A(p)$ generates A and $E = M(p)$ generates M. We will establish that for $n > 0$, $M_{-n} + M_n \subset V^n E$, thus $M(n) \subset V^n E$. The proof for M_{-n} being similar we only have to prove that $M_n \subset V^n E$. Since $M = \cup_{m=0}^{\infty} V^m E$ there is an r such that $M_n \subset V^r E$, that is for every $x \neq 0$ in M_n, $x = \Sigma v_r v_{r-1}. \cdots .v_1 v_0$ with $v_0 \in E$ $v_i \in V, i = 1, \ldots, s$. We may assume that $v_1 v_o \notin E = M(p)$, $v_{i+1} v_i \notin V = A(p)$ since otherwise we may just shorten the decomposition of x. In other words we may assume $\deg(v_{i+1} v_i) > p$ for $i > 0$. Since $\deg(v_r. \cdots .v_1 v_0) = \sum_0^r \deg v_i = n > 0$ it follows that $\deg\, v_i > 0$ for at least one index i. Assuming then $\deg\, v_{i+1} \leq 0$ yields :

$$|\deg(v_{i+1} v_i)| \leq \mathrm{Max}\{(\deg v_i), (\deg v_j)\} \leq p$$

a contradiction. Therefore deg$v_j > 0$ (if $r = i$ consider then $v_r v_{r-1}$, in stead of $v_{i+1}v_i$) and thus $n \geq r+1$, or $v_r v_{r-1}, \ldots, v_1 v_0 \in V^n E$ and $M_n \subset V^n E$. From $d_M(n) \leq \dim_K(V^n E)$, and using 1. statement 2. follows. $\square$

Recall that a filitered A-module M over a filtered ring A with filtration FM, resp. FA, is said to be **finitely filtered** (or FM, resp. FA, is said to be a **finite filtration**) if for every $i \in \mathbb{Z}$, $\dim_K F_i M < \infty$, resp. $\dim_K F_i A < \infty$. The filtration was left limited if $F_m M = 0$ for every $m \leq m_0$ for some $m_0 \in \mathbb{Z}$.

6.2.2 Lemma

With notation as above let A have left limited filtration.

1. If $M_1 \subset M_2$ are submodules of M such that $G(M_1) = G(M_2)$ then $M_1 = M_2$.

2. If $G(M)$ is a left Noetherian $G(A)$-module then M is left Noetherian A-module.

3. $GK\dim_{G(A)} G(M) \leq GK\dim M$.

Proof

1. and 2. have been observed earlier, subsection 4.2.

3. Look at finite dimensional K-subspaces $W \subset G(A), F \subset G(M)$. There exist finite dimensional K-subspaces $V \subset A$ with $1 \in V$, $E \subset M$ such that $W \subset G(V), F \subset G(E)$. Thus : $W^n F \subset G(V)^n G(E) \subset G(V^n)G(E) \subset G(V^n E)$! Indeed, if $\overline{v}_t \in G(V^n)_i, \overline{e}_r \in G(E)_r$ then either $\overline{v}_t \overline{e}_r = 0$ or else $\overline{v}_t \overline{e}_r = v_t e_r \bmod F_{t+v-1}M$ for certain representatives $v_t \in F_t(V^n) e_r \in F_r(E)$, i.e. $\overline{v}_t \overline{e}_r \in G(V^n E)_{t+r}$. Hence for all n we obtain $\dim_K W^n F \leq \dim_K V^n E$, what yields the desired inequality.

$\square$

In the finitely generated situation equality holds in 3. above !

6.2.3 Proposition

Suppose A is finite filtered such that $G(A)$ is a finitely generated K-algebra. Let M be a finite filtered A-module such that $G(M)$ is a finitely generated $G(A)$-module. Put $d_M = \dim_K F_n M$, then : $G(G(M)) = G(M) = G(d_M) = G(d_{G(M)})$. In particular : $GK\dim_{G(A)} G(M) = GK\dim_A M = \overline{\lim} \log_n d_M(n)$.

Proof Recall that : $d_{G(M)}(n) = \dim_K(\oplus_{-n}^n G(M)_n)$, also we write $(FM)(n)$ for $\oplus_{-n}^n \frac{F_j M}{F_{j-1} M}$, as a K-vectorspace. Thus $d_M \sim d_{G(M)}$ and $G(d_M) = (G(d_{G(M)}))$. From Lemma 6.2.1.(2) it follows that $G(G(M) = G(d_{G(M)})$. It is clear (Section 4.3.) that A is a finitely generated K-algebra and M is a finitely generated A-module (lift sets of generators from $G(A)$, resp. $G(M)$, to A, resp. M). Consider $E \subset M, V \subset A$ with $1 \in V$, generating finite dimensional vectorspaces. There exists a $p \in \mathbb{N}$ such that $V \subset A(p), E \subset M(p)$. Thus $V^n E \subset A(p)^n M(p) \subset M(2pn)$ for $n \geq 1$, hence $d_{V.E}(n) \leq d_M(2pn)$ or $G(M) \leq G(d_M)$. From Lemma 6.2.2.(3) the other inequality follows, hence on the level of $GK\dim$ we arrive at : $GK\dim_{G(A)}G(M) = GK\dim_A M = \overline{\lim}\ \log_n d_M(n)$. $\square$

Recall the definition of good filtration on a filtered A-module M (cf. 4.2.6.). Observe that if FA and FM are left limited (discrete filtrations) then FM is good if $G(M)$ is a finitely generated $G(A)$-module.

6.2.4 Lemma

If FA and FM are left limited filtrations then FM is a good filtration if and only if $G(M)$ is a finitely generated $G(A)$-module (if and only if $\widetilde{M}$ is a finitely generated graded $\overline{R}$-module).

Proof Since FM is good there exist $m_1,\ldots,m_d \in M$ such that $F_nM = F_{n-n_1}Am_1 + \ldots + F_{n-n_d}Am_d$ for certain $n_1,\ldots,n_d \in \mathbb{Z}$. Observe that $m_i \in F_{n_i}M$ for $i = 1,\ldots,d$ follows from $F_{n_i}M = F_{m_i-n_i}Am_1 + \ldots + F_0Am_i + \ldots$. Suppose $m_i \in F_{n_i-1}M$, then $m_i \in F_{n_i-1-n_1}Am_1 + \ldots + F_{-1}Am_i + \ldots$ but $F_{-1}A \subset J(F_0A)$ because $(F_{-1}A)^e = 0$ for some $e \in \mathbb{N}$, thus $(1+z)m_i \in F_{n_i-1-n_1}Am_1 + \ldots + F_{n_i\ldots 1-n_d}Am_d$, for some $z \in J(F_0A)$, hence $(1+z)^{-1} \in F_0A$ and thus $m_i \in F_{n_i-1-n_1}Am_1 + \ldots + F_{n_i-1-n_d}Am_d$ where m_i does not appear in the sum on the right. This means that we may delete m_i in the set of generators for M. So if we assume $m_1,\ldots,m_d$ is minimal as a set of generators for M then $m_i \in F_{n_i}M - F_{n_i-1}M$ for every $i = 1,\ldots,d$ and then the $\sigma(m_i)$ are generators for $G(M)$ where σ is the principal symbol map. Conversely if $G(M)$ is finitely generated by $\overline{m}_1,\ldots,\overline{m}_d$, this can be lifted to $m_i \in F_{n_i}M - F_{n_i-1}M$ and a left A-module generated by the $m_i, i = 1,\ldots,d$, say N. Then N is a filtered submodule of M, thus also left limited such that $G(M) = G(N)$. Thus $M = N$ follows and for $m \in F_nM - F_{n-1}M$ we have $\overline{m} \in G(M)_n, \overline{m} = \sum_i \overline{a}_{n-n_i}\overline{m}_i$. Then $m - \sum_i a_{n-n_i}m_i \in F_{n-1}M$ and we now finish the proof by recurrence on n (some $F_{n_0}M = 0$) repeating the foregoing argumentation. We arrive at $m \in \sum_i F_{n-n_i}Am_i$. $\square$

6.2.5 Remark

Finite filtrations are left limited.

6.2.6 Proposition

Let A be finitely filtered by FA and let M be an A-module with finite filtrations F^1M, F^2M. If F^1M is a good filtration then there is an $n \in \mathbb{N}$ such that $F_i^1M \subset F_{i+n}^2M$ for all i. If F^2 is good too then $F^1 \sim F^2$.

Proof In view of the remark we may find a $q \in \mathbb{Z}$ such that $F_iA = 0$, $F_i^1M = F_i^2M = 0$ for $i < -q$. Since F^1M is a good filtration we may apply Lemma 6.2.4. and conclude that $G_{F^1}(M)$ is a finitely generated $G(A)$-module, hence there exists $r \geq q$ such that $G_{F^1}(M)(r) = \oplus_{-r}^r G_{F^1}(M)_j = \oplus_{-q}^r G_{F^1}(M)_j$ is a generating finite dimensional space for $G_{F^1}(M)$. There also exits an n such that $F_r^1M \subset F_{n-q}^2M$. If $-q < i \leq r$ then $F_i^1M \subset F_r^1M \subset F_{n-q}^2M \subset F_{n+i}^2M$. If $i > r$, suppose $F_j^1M \subset F_{j+n}^2M$ for $j < i$, then we have : $G_{F_1}(M)_i = \sum_{j-q}^r G(A)_{i-j}G_{F_1}(M)_j$;

therefore :

$$\begin{aligned} F_i^1 M &\subset \sum_{j=-q}^{r} F_{i-j} A F_j^1(M) + F_{i-1}^1 M \subset \\ &\subset (\sum_{j=-q}^{r} F_{i-j} A F_{j+n}^2 M) + F_{j-1+n}^2 M \\ &\subset F_{i+n}^2 M \end{aligned}$$

So we conclude by induction.
The second statement follows by interchanging F^1 and F^2 in the foregoing proof, then equivalence of F^1 and F^2 follows. □

6.2.7 Theorem

Let A be a K-algebra with finite filtration FA such that $G(A)$ is a finitely generated left Noetherian K-algebra. Then $GK\dim$ is exact on finitely generated left modules.

Proof Consider an exact sequence $0 \to N \to M \to P \to 0$ of finitely generated left A-modules. Let $M = AE$ for some finite dimensional subspace E in M and look at the standard filtration on M, $F_n M = FA_n E$, and the induced filtrations on N and P. So the exact sequence is strict exact. Hence from 3.2., cf. Theorem 3.2.20, we retain that : $0 \to G(N) \to G(M) \to G(P) \to 0$ is an exact sequence of graded $G(A)$-modules. Since $G(A)$ is left Noetherian all these graded modules are finitely generated since $G(M)$ is finitely generated. Now $d_M(n) = \dim_K F_n M = \dim_K (F_n M \cap N) + \dim_K \left(\frac{N+F_nM}{N}\right) = d_N(n) + d_P(n)$. Thus $\overline{\lim}\ \log_n d_M(n) = \mathrm{Max}\{\overline{\lim}\ \log_n d_N(n), \overline{\lim}\ \log_n d_P(n)\}$. From 6.2.3. it follows that :

$$\begin{aligned} GK\dim_A M &= GK\dim_{G(A)} G(M) \\ &= \mathrm{Max}\{GK\dim_{G(A)} G(P), GK\dim_{G(A)} G(N)\} = \\ &= \mathrm{Max}\{GK\dim_A P, GK\dim_A N\} \end{aligned}$$ □

6.2.8 Definition

A K-algebra A is **almost commutative** if A has a filtration FA such that :

i. $F_i A = 0$ for $i < 0$ and $F_0 A = K$, FA is finite.

ii. $A = K[F_1 A]$ as a K-algebra.

iii. $G(A)$ is commutative.

6.2.9 Proposition

Let A be an almost commutative K-algebra.

1. $G(A)$ is finitely generated K-algebra, hence Noetherian.
2. A is Noetherian.

Proof 2. follows from 1. We have $G(A) = K[G(A)_1]$ hence $G(A)$ is finitely generated, therefore Noetherain since it is also commutative. □

6.2.10 Corollary

For almost commutative A we have exactness of $GK\dim$.

We recall here some facts concerning the Hilbert-Samuel polynomial of a commutative algebra. We refer to standard books on commutative algebra for full detail.

6.2.11 Lemma

Let X be a variable over the rational field $\mathbb{Q}$. For $i \in \mathbb{N}$ consider $\begin{pmatrix} X \\ i \end{pmatrix} = \dfrac{X(X-1)\dots(X-i+1)}{i!}$.

1. Consider a function $f : \mathbb{N} \to \mathbb{R}$. The following statements are equivalent :
   i) There exist $a_0, \dots, a_d \in \mathbb{Q}, m \in \mathbb{N}$, such that for all $n \geq m:\ f(n) = \sum_{i=0}^{d} a_i \begin{pmatrix} n \\ i \end{pmatrix}$.
   ii) There exist $a_0, \dots, a_d \in \mathbb{Q}, m \in \mathbb{N}$ such that for all $n \geq m:\ f(n+1) - f(n) = \sum_{i=0}^{d-1} a_{i+1} \begin{pmatrix} n \\ i \end{pmatrix}$.
2. If $p \neq 0$ in $\mathbb{Q}[X]$, say $p(X) = \sum_{i=0}^{d} a_i \begin{pmatrix} X \\ i \end{pmatrix}$, then from $p(n) \in \mathbb{Z}$ for large enough n, it follows that $a_i \in \mathbb{Z}$. If $p(n) \in \mathbb{N}$ and $p(n+1) - p(n) \geq 0$ for large enough n, then $a_d \in \mathbb{N}^+$.

6.2.12 Theorem

Consider $A = K[X_1, \dots, X_r]$ with its usual gradation by total degree, $A = \oplus_{i=0}^{\infty} A_i$. Let $M = \oplus_{i=-q}^{\infty} M_i$ be a finitely generated graded A-module.

1. We have $\dim_K M_n < \infty$ for all n and the map $m \to \dim_K M_m$ is polynomial of degree less than or equal to $r - 1$.
2. For large enough n, $d_M(n) = \sum_{j=-n}^{n} \dim_K M_j$ is a polynomial of degree r with rational coefficients.

6.2.13 Corollary

Let $A = K[A_1]$ be a commutative graded K-algebra with $\dim_K A_1 < \infty$ and let $M = \oplus_{i=-q}^{\infty} M_i$ be a finitely generated graded A-module. For n large enough : $n \mapsto d_M(n) = \dim_K M(n)$ is a polynomial function with rational coefficients and degree exactly $GK\dim_A M$.

Proof We write A as quotient of $K[X_1, \ldots, X_r]$ with $r = \dim_K A_1$ and we may look at M as a $K[X_1, \ldots, X_r]$-module which is graded. Foregoing results then apply to M as a graded $K[X_1, \ldots, X_r]$-module. Since we have that $GK\dim_A M = GK\dim_{K[X_1,\ldots,X_r]} M$ (see Proposition 6.1.18(3) and since $GK\dim_A M = \overline{\lim}\ \log_n d_M(n)$ (Lemma 6.2.1(2)) the statement about the degree is clear. $\square$

6.2.14 Proposition

Let A be an almost commutative K-algebra and M a good filtered A-module with filtration FM. For n large enough :

$$d_M(n) = \dim_K F_n M = \dim(\sum_{j=-n}^{n} G(M)_j) = d_{G(M)}(n), \text{is polynomial in } n$$

We write : $d_{FM}(n) = a_d \begin{pmatrix} n \\ d \end{pmatrix} + a_{d-1} \begin{pmatrix} n \\ d-1 \end{pmatrix} + \ldots + a_1 \begin{pmatrix} n \\ 1 \end{pmatrix} + a_0$,where $d = GK\dim_A M \in \mathbb{N}$ and $e_{FM}(M) = a_d$ is called the **multiplicity or Bernstein number** of M, $a_d \geq 1$ in $\mathbb{N}$

Proof Observe that $\dim_K F_n M < \infty$ since $G(M)$ is finitely generated over $G(A)$. We apply foregoing results taking into account that $GK\dim_A M = GK\dim_{G(A)} G(M)$. Observe $d_{FM}(n) = \frac{e(M)}{d!} n^d + \ldots$, for n large enough. $\square$

6.2.15 Proposition

Let A be an almost commutative K-algebra M a filtered A-module with good filtrations $F^1 M$ and $F^2 M$, then $e_1(M) = e_2(M)$, for $e_i = e_{F^i M}, i = 1, 2$.

Proof We know that $F^1 M$ and $F^2 M$ are equivalent filtrations (Proposition 6.2.6) hence there is a $q \in \mathbb{N}$ such that $d_{F^2 M}(n) \leq d_{F^1 M}(n + q)$. This entails : $e_2(M) \leq e_1(M)$. The other inequality follows by interchanging F^1 and F^2. Observe that $e(M)$ does depend on the filtration FA, but as proved before not on the choice of good filtration on M.

6.2.16 Theorem

Consider an almost commutattive K-algebra A and an exact sequence of finitely generated left A-modules : $0 \to L \to M \to N \to 0$.

1. $GK\dim_A M = \mathrm{Max}\{GK\dim_A L, GK\dim_A N\}$, i.e. $GK\dim$ is exact.

2. One of the following assertions holds :

   a. $GK\dim_A L < GK\dim_A M, GK\dim_A M = GK\dim_A N, e(M) = e(N)$
   b. $GK\dim_A N < GK\dim_A M, GK\dim_A M = GK\dim_A L, e(M) = e(L)$
   c. $GK\dim_A L = GK\dim_A M = GK\dim_A N$ and $e(M) = e(L) + e(N)$

Proof

1. Follows from Proposition 6.2.7.

2. Look at a standard filtration M, $F_nM = F_nAE$ for some generating subspace E and put the induced filtrations on L and N. As in the proof of 6.2.7. we find : $d_{FL}(n) = d_{FN}(n) = d_{FM}(n)$. The statements follow from Proposition 6.2.14. □

Observe that, under the condition of foregoing theorem we may define standard filtrations on L, M, N, such that $0 \to G(L) \to G(M) \to G(N) \to 0$ is exact.

6.2.17 Corollary

Let A be almost commutative and M a finitely generated left A-module with $GK\dim_A M = d$ and Bernstein number e. Consider in M a descending chain of A-submodules : $M = M_0 \supsetneqq M_1 \supsetneqq \dots \supsetneqq M_n$, such that $GK\dim_A(M_i/M_{i+1}) = d$ for $i, 0 \le i \le n-1$, then we have $e(M/M_i) = \sum_{j=0}^{i-1} e(M_j/M_{j+1})$ and $n \le e$.

Proof The first statement follows by recurrence applying the foregoing theorem. The second statement follows from the first because

$n \le \sum_{j=0}^{n-1} e(M_j/M_{j+1}) = e(M/M_n) \le e$ □

This corollary has an important consequence. If M as in the corollary has $GK\dim_A M = d$ and we look at a descending chain $M = M_0 \supset M_1 \supset \dots \supset \dots$ of A-submodules then $GK\dim M_i = d$ holds for all i and by the corollary this d.c. must become stationary as it has at most $e(M)$ entries that are different. Hence M is also left Artinian. Now a left Artinian left Noetherian module has finite length i.e. there is a finite composition series $M = M_0 \supset M_1 \supset \dots \supset M_l = 0$ where each M_j/M_{j+1} is a simple A-module, $j \in \{0, \dots, l-1\}$. The number l is called the **length of** M and it does not depend on the chosen composition series. Hence $l(M) \le e$. Let M have good filtration over an almost commutative K-algebra A. We define $I_{FM}(M) = \{x \in G(A), xG_F(M) = 0\}$. The prime radical of $I_{FM}(M)$, denoted by $N(I_{FM}(M)) = \mathcal{N}_F(M)$, is called the **characteristic ideal of** M.

6.2.18 Lemma

If F^2M is another good filtration of M, then $\mathcal{N}_F(M) = \mathcal{N}_{F^2}(M)$.

Proof We know that FM and F^2M are equivalent filtrations, hence there is an $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$:

$$F_nM \subset F^2_{n+n_0}N, F^2_nM \subset F_{n+n_0}M$$

In view of the symmetry between F and F^2 it suffices to show that $\mathcal{N}_F(M) \subset \mathcal{N}_{F^2}(M)$. Since we are dealing with graded ideals it suffices to establish that an $\overline{a}_p \in G(A)_p$ in $I_{FM}(M)$ is necessarily contained in $\mathcal{N}_{F^2}(M)$. For a_p representing $\overline{a}_p$ it follows that $a_pF_nM \subset F_{n+p-1}M$ for all n, hence for all $q, n \in \mathbb{N}$ we have : $a^qF_nM \subset F_{n+qp-q}M$. In particular for $q = 2n_0+1$ we have the following :$a^qF_n^2M \subset a^qF_{n+n_0}M \subset F_{n+n_0+qp-2n_0}M \subset F^2_{n+pq-1}M$. Hence $\overline{a}^qG_{F^2}(M)_n = 0$ or $\overline{a}^q \in I_{F^2M}(M)$, thus $\overline{a} \in \mathcal{N}_{F^2}(M)$. □

6.2.19 Proposition

Let A be an almost commutative K-algebra.

1. If M is a finitely generated left A-module then we have : $GK\dim_A M = K\dim(G(A)/\mathcal{N}_F(M))$.
2. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated left A-modules than : $\mathcal{N}_F(M) = \mathcal{N}_F(L) \cap \mathcal{N}_F(N)$.

Proof

1. If FM is a good filtration, then we obtain :

   $$\begin{aligned} GK\dim_A M = GK\dim_{G(A)}G(M) = GK\dim_{G(A)}(G(A)/I_{FM}(M)) \\ = GK\dim_{G(A)}(G(A)/\mathcal{N}_F(M)) \end{aligned}$$

   Since $G(A)$ is commutative :

   $GL\dim_{G(A)}(G(A)/\mathcal{N}_F(M)) = K\dim(G(A)/N_F(M))$.

2. Choose good filtrations on L, M, N such that the sequence $0 \to G(L) \to G(M) \to G(N) \to 0$ is exact. Then we have $I_{FM}(M) = I_{FL}(L) \cap I_{FN}(N)$ and, by taking the prime radical, the statement follows. □

If K is algebraically closed and A is an almost commutative K-algebra then $G(A)$ is a quotient of $K[X_1, \dots, X_n]$, say $G(A) = K[X_1, \dots, X_n]/I$. If M is a finitely generated left A-module with characteristic ideal $N_F(M)$ then $G(A)/\mathcal{N}_F(M)$ is isomorphic to a quotient of $K[X_1, \dots, X_n]$ with respect to a radical ideal of $K[X_1, \dots, X_n]$ again denoted $\mathcal{N}(M)$.

From the Hilbert nulstellen-satz every radical ideal of $K[X_1, \dots, X_n]$ corresponds to an algebraic set in $K^n(= \mathbb{A}_n(K)$ called the **characteristic variety** of M :

$$V(M) = \{(x_1, \dots, x_n) \in K^n, f(x_1, \dots, x_n) = 0 \text{ for all } f \in \mathcal{N}(M)\}$$

Up to isomorphism $V(M)$ does not depend on the presentation of $G(A)(= k[X_1, \dots, X_n]/I)$.

We finish this section with some applications to the module theory of Weyl algebras.

If M is a nonzero module over $\mathbb{A}_1(K)$, then $GK\dim M \geq 1$. Indeed, $GK\dim M$ is an integer, if $GK\dim M = 0$, then $\dim_K M < \infty$, but we have observed that there are no finite dimensional (over K) $\mathbb{A}_1(K)$-modules. Now we want to establish **Bernstein's inequality** for $\mathbb{A}_n(K)$, i.e. $GK\dim M \geq n$ for every nonzero $\mathbb{A}_n(K)$-module M.

6.2.20 Lemma

Let $A \subset B$ be almost commutative K-algebras with standard filtrations given by finite dimensional generating spaces $V \subset W$ resp. Let M be a finitely generated left B-module with $GK\dim_B M = d$ and Bernstein number $e_B(M)$. Then for every finitely generated left A-submodule N of M we have :

a. $GK\dim_B N \leq d$

b. If $GK\dim_A N = d$, then $e_A(N) \leq e_B(N)$.

Proof

a. Consider $N_0 \subset M_0$ finite dimensional generating K-spaces such that $N = AN_0, M = BM_0$. Define FN, FM by putting $F_n N = V^n N_0, F_n M = W^n M_0$, then $F_n N \subset F_n M$. Then we obtain :

   $GK\dim_A N \leq \overline{\lim} \log_n \dim_K F_n M$, the latter is exactly $GK\dim_B M = d$.

b. If $GK\dim_A N = d = GK\dim_B M$, then for almost all n (n large enough) : $\dim_K F_n M = \frac{e_B(M)}{d!} n^d + \ldots$ is larger than $\dim_K F_n N = \frac{e_A(N)}{d!} n^d + \ldots$. Consequently $e_B(M) \geq e_A(N)$ follows, as claimed. $\square$

6.2.21 Theorem (Bernstein inequality)

If M is a nonzero left $\mathbb{A}_n(K)$-module, then $GK\dim M \geq n$.

Proof Write $\mathbb{A}_n(K) = K < x_1, \ldots, x_n, y_1, \ldots y_n >$. The proof is by recurrence on n. The case $n = 1$ is easy, in fact we proved this in the comments preceding Lemma 6.2.20. Now look at $\mathbb{A}_n(K) = \mathbb{A}_{n-1}(K) < x_n, y_n >$ and put $A = \mathbb{A}_{n-1}(K), B = \mathbb{A}_n(K)$. Look at a nonzero left B-module and assume $GK\dim_B M < n$. Consider a nonzero A-submodule N in M. By the induction hypothesis it follows that : $n - 1 \leq GK\dim_A N \leq GK\dim_A M \leq GK\dim_B M < n$ (using a. from the lemma). Consequently : $n - 1 = GK\dim_A N = GK\dim_B M$. From the statement b. in the lemma we then obtain $e_A(N) \leq e_B(M)$.

Since every nonzero submodule N has $GK\dim_A N \geq n - 1$ it follows from Corollary 6.2.17 that N has a composition series : $N_0 = N \underset{\neq}{\supseteq} N_1 \underset{\neq}{\supseteq} \ldots \underset{\neq}{\supseteq} N_p = 0$, with N_i/N_{i+1} simple for $0 \leq i \leq p - 1$, and $p \leq e_A(N) \leq e_B(M)$. The foregoing holds for every finitely generated A-module contained in M and thus as a left A-module M has finite length less that $e_B(M)$. We will prove hereafter (Quillen's lemma) that it follows that $\mathrm{End}_A(M)$ is algebraic over K. Since the elements of $C = K < x_n, y_n >$ commute with A, multiplication by y_n in M yields an element of $\mathrm{End}_A(M)$ thus there exists a polynomial $f \neq 0$ such that $f(y_n) = 0$. Take an f which is minimal with respect to this property. From $[x_n, f(y_n)] = f'(y_n), f'$ being the derivation of f, it follows that $f'(y_n) = 0$ in $\mathrm{End}_A(M)$ and thus $f' = 0$ because of the minimality assumption. Since we assumed $ch(K) = 0$ throughout this section, we have $f =$ constant, contradicting $f(y_n) = 0$. Hence we must have $GK\dim_B M \geq n$. $\square$

For a simple module M over a ring A we call $\text{End}_A(M)$ the **commutant** of M. From Schur's Lemma we retain that for a simple A-module M, $\text{End}_A(M)$ is a skewfield. Recall the following result from commutative algebra.

6.2.22 Lemma

Let R be a commutative domain and S a finitely generated R-algebra that is also commutative. Let N be a finitely generated left S-module. Then there exists an $x \neq 0$ in R such that the localized module N_x at $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a free left R_x-module.

6.2.23 Theorem (Quillen's Lemma)

Let A be a filtered K-algebra such that $G(A)$ is a finitely generated commutative K-algebra. Let M be a simple left A-module with commutant D, then every element of D is algebraic over K.

Proof Take $x \in D, R = K[x] \subset D$ and suppose that x is not algebraic over K. Put $B = A \otimes_K K[x] = A[x]$. Now M is a B-module via : $a.m = am, x.m = x(m)$ for $a \in A$, $m \in M$ (observe that : $(ax).m = a.x(m) = x(am)$). If $m_0 \neq 0$ in M then we have $M = Am_0$ and thus $M = Bm_0$. We define a filtration on B and M as follows :

$$F_pB = F_pA \otimes_K K[x], F_pM = F_pB.m_0$$

Then M is a filtered B-module. Now $G(B) = \oplus_{p \geq 0} F_pB/F_{p-1}B$ contains $R = K[x]$ if we identify x with its class in $G(B)_0 = F_0B = R$. It is easily verified that a finite set of generators of $G(A)$ yields a finite set of generators of $G(B)$ and that $G(B)$ is commutative. On the other hand $G(M) = \oplus_{p \geq 0} F_pM/F_{p-1}M$ and this is generated over $G(B)$ by $(m_0)_0 \in F_0M/F_{-1}M = F_0M$. We now apply the lemma with $S = G(B), R$ and $N = G(M)$, so we find an $f \in R - \{0\}$ such that $G(M)_f$ is a free left R_f-module; observe that $F_pM/F_{p-1}M$ is a $G(B)_0 = R$-module and thus $G(M)$ is an R-module too. Since $R = K[x]$ is a principal ideal ring also R_f is a principal ideal ring and hence all $(F_pM/F_{p-1}M)_f$ are free left R_f-modules. We have that $M_f = \cup_{p \geq 0} (F_pM)_f$ and therefore M_f is a free left R_f-module (exercise !). If we take a nonzero g on R which is not a divisor of f^t, $t \geq 0$, then multipication by g in R_f is not surjective, indeed if it were surjective then $1 = g.\frac{a}{f^t}$ for some $t \geq 0$ would entail $f^t = ga$ or g would be a divisor of f^t. Since M_f is free as a left R_f-module then $\mu \in \text{End}_{A_t}(M_f)$ defined by $\mu(z) = g.z$ is not surjective. For $y \in M_f$, say $y = f^{-t}\alpha$ for some $\alpha \in M, t \in \mathbb{N}$. Since $0 \neq g \in R \subset D$ and D is a field, g must define a bijection on M, that is $\alpha = g.\beta$ for certain $\beta \in M$. Then we obtain : $y = f^{-t}, (g.\beta) = g, (f^{-t}.\beta) = \mu(f^{-t}.\beta)$, but this contradicts the non-surjectivity of μ. $\square$

In fact, we needed in Theorem 6.2.21 the following generalization of the foregoing Theorem.

6.2.24 Corollary

Let A be as in the foregoing theorem and let M be a left A-module of finite length. Then the commutant of M is algebraic over K.

Proof Write the length of M as $l(M) = p$ and look at a composition series of length p : $M_0 = M \underset{\neq}{\supseteq} M_1 \underset{\neq}{\supseteq} \ldots \underset{\neq}{\supseteq} M_p = 0$. We argue by recurrence on p. The case $p = 1$ has been proven before, so we look at the situation $p > 1$. Choose any $x \in \mathrm{End}_A(M)$. If Ker$x \neq 0$, put $\overline{M} = M\mathrm{Ker}\ x$, then $l(\overline{M}) < l(M)$ and $\overline{x} \in \mathrm{End}_A(\overline{M})$ is defined by $\overline{x}(\overline{m}) = \overline{x(m)}$. Thus there exists an $f \neq 0$ such that $f(\overline{x})(\overline{m}) = 0$ for all $\overline{m} \in \overline{M}$, in other words $f(x)(m) \in \mathrm{Ker}(x)$ for all $m \in M$. Therefore in $\mathrm{End}_A(M)$ we have $xf(x) = 0$. In case $\mathrm{Ker}(x) = 0$ then $\mathrm{Ker}(x^t) = 0$ for all $t \geq 1$ and this entails $S \cong x^t S$ for every simple left A-module S, in particular $x^t S$ is again a simple left A-module. Now consider a simple left A-module of M, say S.

- *Case 1*

  $S \cap xS \neq 0$. Then $S = xS$. Define $\overline{x}$ in $\mathrm{End}_A(M/S)$ by putting $\overline{x}(m + S) = x(m) + S$. Since $l(\overline{M}) < l(M)$ there exists a polynomial $f \neq 0$ such that $f(\overline{x}) = 0$. Because S is simple, the theorem entails the existence of a nonzero polynomial g such that $g(x) = 0$ in $\mathrm{End}_A(S)$. Thus $f(x)M \subset S$ and $g(x)f(x)M = 0$ with $gf \neq 0$.

- *Case 2.*

  $S \cap xS = 0$ (still assuming $\mathrm{Ker}(x) = 0$). We establish by recurrence on t the validity of the following :

  $(*)$ For all $t \in \mathbb{N}$, for all $n_1 < n_2 < \ldots < n_{t+1}$:
  $$(x^{n_1}(S) + \ldots + x^{n_t}(S)) \cap x^{n_{t+1}}(S) = 0$$

  If (*) holds, then $\sum_{i=0}^{\infty} x^n(S)$ is a direct sum in M but that would contradict the assumption that $l(M) < \infty$, hence we are in the situation $S \cap x(S) \neq 0$ so that the statement follows because $gf \neq 0$ yields a polynomial having x as a solution. So we finish the proof by :

  Proof of (*) If $t = 1$ and $x^{n_1}(s_1) = x^{n_2}(s_2)$ for some s_1 and s_2 in S, then we obtain (since $n_2 > n_1$) :
  $$x^{n_1}(x^{n_2 - n_1}(s_2) - s_1) = 0$$
  hence $s_1 \in S \cap x(S) = 0$. If $\sum_{i=1}^{t} x^{n_i}(s_i) = x^{n_{t+1}}(s_{t+1})$, the we may assume $n_1 = 0$ because $\mathrm{Ker}(x^{n_1}) = 0$, thus :
  $$s_1 = x^{n_2}(x^{n_{t+1} - n_2}(s_{t+1})) - \sum_{i=2}^{t} x^{n_i - n_2}(s_i)$$
  The latter is in $S \cap xS$ because $n_2 > 0$. Therefore $s_1 = 0$ and by recurrence $x^{n_j}(s_j) = 0$ follows for all j. $\square$

Chapter 7

Global Dimension of Filtered Rings

We introduce briefly some notions concerning projective dimension of modules and global dimension of rings. For full detail on general homological algebra we refer to the book of J. Rotman, [20], also [19]. Homological algebra is an important tool both in commutative algebra, algebraic geometry as well as in noncommutative algebra and recent noncommutatie geometry.

7.1 Projective Resolutions and Homological Algebra

Over a ring R a **complex** $\dot{M}$ is a sequence of maps in R-mod :

$$\ldots \to M_n \xrightarrow[d_n]{} M_{n-1} \to \ldots, n \in \mathbb{Z}$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The maps d_n are called **differentiations**. A **chain map** $f : \dot{M} \to \dot{M}^1$ is a family of homomorphisms $f_n : M_n \to M_n^1$, $n \in \mathbb{Z}$, making the following diagram a commutative one :

$$\begin{array}{ccccccc}
\cdots \longrightarrow & M_n & \xrightarrow{d} & M_{n-1} & \xrightarrow{d} & \cdots \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1a}} & & \\
\cdots \longrightarrow & M_n^1 & \xrightarrow[d^1]{} & M_{n-1}^1 & \longrightarrow & \cdots
\end{array}$$

If $\dot{M}$ is a complex then $\mathrm{Im} d_{n+1} \subset \mathrm{Ker} d_n$ and the n^{th} **homology group** $H_n(\dot{M})$ is defined as $\mathrm{Ker} d_n/\mathrm{Im} d_{n+1}$. Elements of Ker_n are n-**cycles**, elements of $\mathrm{Im} d_{n+1}$ are n-**boundaries**. We put $Z_n(\dot{M}) = \mathrm{Ker} d_n$, $B_n(\dot{M}) = \mathrm{Im} d_{n+1}$, $H_n(\dot{M}) = Z_n(\dot{M})/B_n(\dot{M})$. If $f : \dot{M} \to \dot{M}^1$ is a chain map, then $H_n(f) : H_n(\dot{M}) \to H_n(\dot{M}^1)$ is given by $z_n + B_n(\dot{M}) \mapsto f_n(z_n) + B_n(\dot{M}^1)$. Clearly $\dot{M}$ is exact if and only if $H_n(\dot{M}) = 0$ for all $n \in \mathbb{Z}$. If $0 \to \dot{M}' \to \dot{M} \to \dot{M}'' \to 0$ is an exact sequence of complexes (i.e. $0 \to M_n' \xrightarrow{\iota} M_n \xrightarrow{\pi} M_n'' \to 0$ is exact for all n), then for every n there is a morphism ∂_n, $\partial_n : H_n(\dot{M}'') \to H_{n-1}(\dot{M}')$, given by : $z'' + B_n(\dot{M}'') \mapsto \iota^{-1} d\pi^{-1}(z'') + B_{n-1}(\dot{M}')$. This can be checked by a routine diagram - chase argument. The morphisms δ_n are the **connecting morphisms**.

7.1.1 Theorem (The Exact Triangle)

If we have an exact sequence of complexes : $0 \to \dot{M}' \to \dot{M} \to \dot{M}'' \to 0$, then there is an exact sequence :

$$\ldots \to H_n(\dot{M}) \underset{H_n(\pi)}{\longrightarrow} H_n(\dot{M}'') \underset{\delta_n}{\longrightarrow} H_{n-1}(\dot{M}') \underset{H_{n-1}(\iota)}{\longrightarrow} H_{n-1}(\dot{M}) \to \ldots$$

Proof

1. $\mathrm{Im} H_n(\iota) \subset \mathrm{Ker} H_n(\pi)$. This follows from
$$H_n(\pi) \circ H_n(\iota) = H_n(\pi \circ \iota) = H_n(0) = 0$$
2. $\mathrm{Ker} H_n(\pi) \subset \mathrm{Im} H_n(\iota)$. If $H_n(\pi)(r_n + B_n(\dot{M})) = \pi_n(z_n) + B_n(\dot{M}'') = B_n(\dot{M}'')$, then $\pi_n(z_n) = \partial''_{n+1}(a''_{n+1})$ but $a''_{n+1} = \pi_{n+1}(a)_{n+1}$ for some $a_{n+1} \in M_{n+1}$. Hence $\pi_n(z_n)$ equals $\partial''_{n+1}(\pi_{n+1}(a_{n+1})) = \pi_n \partial_{n+1}(a_{n+1})$, thus $z_n - \partial_{n+1}(a_{n+1}) \in \mathrm{Ker}\pi_n$.

   By exactness there is an $a' \in A'_n$ such that $\iota_n(a') = z_n - \partial_{n+1}(a_{n+1})$. Now $\iota_{n+1}\partial'_n(a') = \partial_n \iota_n(a') = \partial_n(z_n) - \partial_n \partial_{n+1}(a_{n+1}) = \partial_n(z_n) = 0$ (the latter because z_n is a cycle). Since ι is monic it follows that $\partial'_n(a') = 0$, therefore $H_n(\iota)(a' + B_n(\dot{M}') = \iota_n a' + B_n(\dot{M}) = z_n - \partial_{n-1}(a_{n+1}) + B_n(\dot{M}) = z + B_n(\dot{M})$.
3. $\mathrm{Im} H_n(\pi) \subset \mathrm{Ker}\partial_n$. This follows from : $\partial_n H_n(\pi)(z_n + B_n(\dot{M})) = \partial_n(\pi_n(z_n) + B_n(\dot{M}'')) = x'_{n-1} + B_{n-1}(\dot{M}')$, where $\iota_{n-1} x'_{n-1} = \partial_n \pi_n^{-1} \pi_n(z_n) = \partial_n z_n = 0$ (using that ∂ is well defined).
4. $\mathrm{Ker}\partial_n \subset \mathrm{Im} H_n(\pi)$. Take $z''_n + B_n(\dot{M}'') \in \mathrm{Ker}\partial_n$, then $\partial_n(z''_n + B_n(\dot{M}'') = B_n(\dot{M}')$ yields $x'_{n-1} = \iota_{n-1}^{-1} d_n \pi^{-1}(z''_n) \in B_{n-1}(\dot{M}')$. Hence $x'_{n-1} = \delta'_n(a'_n)$, for some $a'_n \in M'_n$. Now $\iota_{n-1}(x'_{n-1}) = \iota_{n-1} d_n(a'_n) = d_n \iota_n(a'_n) = d_n \pi_n^{-1}(a''_n)$ so that $d_n(\pi_n^{-1}(a''_n) - \iota_n a'_n) = 0$, this means that $\pi_n^{-1}(a''_n) - \iota_n a'_n \in Z_n(M_n)$. Therefore we obtain $H_n(\pi)(\pi_n^{-1}(a''_n) - \iota_n(a'_n) + B_n(\dot{M})) = \pi_n \pi_n^{-1}(a''_n) - \pi_n \iota_n(a'_n) + B_n(M'') = a''_n + B_n(M'')$.
5. $\mathrm{Im}\partial_n \subset \mathrm{Ker} H_{n-1}(\iota)$. We have that $\iota_{n-1}\partial_n(x''_n + B_n(M'')) = \iota x'_{n-1} + B_{n-1}(M)$, with $\iota x'_{n-1} = d_n \pi_n^{-1}(x''_n) \in B_{n-1}(M)$, hence $\delta_n(x''_n + B_n(M'') = 0$.
6. $\mathrm{Ker} H_{n-1}(\iota) \subset \mathrm{Im}\partial_n$. Suppose that $\iota_{n-1}(z'_{n-1} + B_{n-1}(M')) = i_{n-1}(z'_{n-1}) + B_{n-1}(M) = B_{n-1}(M)$, thus $\iota_{n-1}(z'_{n-1}) = d_n(a_n)$ for some $a_n \in M_n$. Then $d''_n \pi_n(a_n) = \pi_{n-1} d_n(a_n) = \pi_{n-1}(\iota_{n-1}(z'_{n-1})) = 0$ yields $\pi_n(a_n) \in Z(M'')$. But $\partial_n(\pi_n(a_n) + B_n(M'')) = x'_{n-1} + B_{n-1}(M')$, where $\iota x'_{n-1} = d_n \pi_n^{-1} \pi_n(a_n) = d_n(a_n) = \iota z'_{n-1}$. Since ι is a monomorphism $x'_{n-1} = z'_{n-1}$ and thus $\partial_n(\pi_n(a_n) + B_n(M'')) = x'_{n-1} + B_{n-1}(M') = z'_{n-1} + B_{n-1}(M')$, or the latter is in the $\mathrm{Im}\partial_n$. □

If we have a chain map $f : \dot{M} \to \dot{M}'$ then we say that f is **nullhomotopic** if there are maps $s_n : M_n \to M'_{n+1}$ such that : $f_n = d'_{n+1} s_n + s_{n-1} d_n$ for all $n \in \mathbb{Z}$. If f and g are chain maps from $\dot{M}$ to $\dot{M}'$ then we say that f **is homotopic to** g if $f - g$ is nullhomotopic.

7.1.2 Proposition

If f and g are homotopic chain maps $\dot{M} \to \dot{M}'$ then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.

Proof If $z_n \in B_n(\dot{M})$ then $f_n(z_n) - g_n(z_n) = d'_{n+1}s_n(z_n) + s_{n-1}d_n(z_n)$. Since $d_n(z_n) = 0$ we arrive at $f_n(z_n) - g_n(z_n) \in B_n(M')$ and therefore $H_n(f) = H_n(g)$.

□

If we have a complex $\ldots \to P_n \to P_{n-z} \to - \to P_0 \to M$ then the **deleted complex** is defined as : $\ldots \to P_1 \to P_0 \to 0$.

7.1.3 Theorem (The comparison theorem)

Consider the following diagram in R-mod.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"'] & P_1 \arrow[r, "d_1"'] & P_0 \arrow[r, "\pi"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & X_2 \arrow[r, "d'_2"'] & X_1 \arrow[r, "d'_1"'] & X_0 \arrow[r, "\pi'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

where both rows are complexes, each P_i is a projective (left) R-module. If the bottom row is exact then there is a chain map of the deleted complexes $\dot{P} \to \dot{X}$ making

$$\begin{tikzcd}
P_0 \arrow[r, "\pi"'] \arrow[d, "\dot{f}_0"'] & M \arrow[d, "f"] \\
X_0 \arrow[r, "\pi'"'] & M'
\end{tikzcd}$$

into a commutative diagram. Two such chain maps are necessarily homotropic.

Proof A somewhat technical "lifting" argument, see [J. Rotman].

We now are ready to define the **left derived functor** for some functor $T : R\text{-mod} \to R\text{-mod}$. For each R-module M choose a projective resolution of M and let $\dot{P}_n$ be the deleted complex : $\ldots P_n \to \ldots \to P_1 \to P_0 \to 0$. Applying T to $\dot{P}_M$ we find the complex : $\ldots \to TP_n \to \ldots \to TP_1 \to TP_0 \to 0$.

7.1.4 Definition

For an R-module M we define the left derived functors L_nT by $(L_nT)M = H_n(T\dot{P}_M)$. If we have an R-linear $M \to N$ then by the comparison theorem there is a chain map $\overline{f} : \dot{P}_M \to \dot{P}_N$ over $f : M \to N$, define $(L_nT)f : (L_nT)M \to (L_nT)N$ by $(L_nT)f = H_n(T\overline{f})$, i.e. if $z_n \in \mathrm{Ker}Td_n$ then $z_n + \mathrm{Im}Td_{n+1} \mapsto (T\overline{f}_n)(z_n) + \mathrm{Im}Td'_{n+1}$. By the comparison theorem $L_nT(f)$ is well-defined because if $g : \dot{P}_M \to \dot{P}_N$ is another chain map over f then $\overline{f}$ and g are homotopic, thus $T\overline{f}$ and Tg are homotopic and so the homology groups are the same.

7.1.5 Example

Let N be a left R-module and put $T = - \otimes_R N$. Then $\mathrm{Tor}_n^R(-, N) = L_nT$, i.e. for all n :

$$\mathrm{Tor}_n^R(M, U) = \mathrm{Ker}(d_n \otimes 1)/\mathrm{Im}(d_{n+1} \otimes 1)$$

where : $\ldots P_2 \xrightarrow[d_2]{} P_1 \xrightarrow[d_1]{} P_0 \to A \to 0$ is a fixed projective resolution for M.

7.1.6 Proposition

The groups $L_nT(M)$ are not depending on the chosen projective resolution for M.

Proof See [J-R], Theorem 6.7. p. 128. $\square$

Right derived functors may be defined by using injective resulotions :

$$0 \to M \to E_0 \underset{d_0}{\longrightarrow} E_{-1} \underset{d_{-1}}{\longrightarrow} E_{-2} \to \dots$$

where each $E_i, 0 \geq i$, is an injective R-module. If T is covariant define the right derived functors R^nT by $R^nT(M) = H_{-n}(T\dot{E}_M) = \mathrm{Ker}(Td_{-n})/\mathrm{Im}(Td_{-n+1})$, where $\dot{E}_M$ is the deleted complex $0 \to E_0 \to E_{-1} \to E_{-2} \to \dots$. If $T = \mathrm{Hom}_R(N, -)$ then $R^nT = \mathrm{Ext}^n_R(N, -)$, this is again independent on the choice of injective resolution ([20], Cor 69).

If T **is contravariant** : look at projective resolutions

$$\dots \to P_n \longrightarrow \dots P_1 \to P_0 \to M \to 0$$

then we arrive at

$$0 \to TP_0 \to TP_1 \to \dots TP_n \to \dots$$

for the deleted resolution $\dot{P}_M$. Rewrite TP_n as TP_{-n} and $\Delta_{-n} : TP_{-n} \to TP_{-n-1}$ for Td_{n+1}. Then define R^nT by

$$R^nT(M) = H_{-n}(T\dot{P}_M) = \mathrm{Ker}(\Delta_{-n})/\mathrm{Im}(\Delta_{-n+1}) = \mathrm{Ker}(Td_{n+1}/\mathrm{Im}(Td_n)$$

In case $T = \mathrm{Hom}_R(-, M)$ then

$$R^nT = \mathrm{Ext}^n_R(-, M), \mathrm{Ext}^n_R(N, M) =$$
$$= \mathrm{Ker}\ \mathrm{Hom}(d_{n+1}, M)/\mathrm{Im}\ \mathrm{Hom}(d_n, M) \text{ for } n \in \mathbb{Z}$$

Again the definition of $\mathrm{Ext}^n_R(N, M)$ is not depending on the chosen projective resolution of M. It is true that the value of $\mathrm{Ext}^n_R(-, M)$ on N equals the value of $\mathrm{Ext}^n_R(N, -)$ on M (see [20] Theorem 6.17, p. (41)).

7.1.7 Theorem

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If T is covariant then there is an exact sequence

$$\dots \to L_nTM' \to L_nTM \to L_nTM''_\gamma \to L_{n-1}TM' \to \dots$$

Also there is an exact sequence :

$$\dots \to R^nTM' \to R^nTM \to R^nTM'' \underset{\partial}{\longrightarrow} R^{n+1}TM' \to \dots$$

In case T is contravariant there is an exact sequence :

$$\dots \to R^nTM'' \to R^nTM \to R^nTM' \to R^{n+1}TM'' \to \dots$$

7.1.8 Theorem

1. If n is negative then $\mathrm{Ext}^n_R(M, N) = 0$ for all M, N.

2. $\mathrm{Ext}^o_R(M, -)$ is naturally equivalent to $\mathrm{Hom}_R(M, -)$.

Proof

1. If $\dot{E}_N$ is
$$\ldots \to 0 \underset{d_2}{\longrightarrow} 0 \underset{d_1}{\longrightarrow} E_0 \underset{d_0}{\longrightarrow} E_{-1} \underset{d_{-1}}{\longrightarrow} E_{-2} \to \ldots$$
then $\mathrm{Hom}_R(M, \dot{E}_N)$ has zeros to the left of $\mathrm{Hom}_R(M, E_0)$ hence all negative homology groups are zero.

2. If $\dot{E}_N$ is as in 1., then $\mathrm{Ext}^n_R(M, N) = \mathrm{Ker\ Hom}(d_0, -)/\mathrm{Im\ Hom}(d_1, -)$. If $0 \to N \underset{\pi}{\longrightarrow} E_0 \to E_{-1} \to \ldots$ is the full resolution then left exactness of $\mathrm{Hom}_R(M, -)$ yields the axat sequence
$$0 \to \mathrm{Hom}_R(M, N) \underset{\mathrm{Hom}(M,\pi)}{\longrightarrow} \mathrm{Hom}_R(M, E_0) \underset{\mathrm{Hom}(M,d_0)}{\longrightarrow} \mathrm{Hom}_R(M, E_{-1})$$
so that $\mathrm{Ker\ Hom}(M, d_0) = \mathrm{Im\ Hom}(M, \pi)$, thus $\mathrm{Hom}(M, \pi)$ is an isomorphism, $\mathrm{Hom}_R(M, N) \cong \mathrm{Ext}^0_R(M, N)$. The natural equivalence of the functors $\mathrm{Ext}^0_R(M, -)$ and $\mathrm{Hom}_R(M, -)$ follows easily from this. □

From Theorem 7.1.7. it follows that from an exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules we obtain the (infinite) long exact sequence :

$$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \xrightarrow{\partial} \mathrm{Ext}^1_R(M, N') \to \ldots$$

7.1.9 Theorem

1. $\mathrm{Ext}^0_R(-, N)$ is naturally equivalent to $\mathrm{Hom}_R(-, N)$.

2. To an exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ there corresponds an exact sequence :
$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \underset{\partial}{\longrightarrow} \mathrm{Ext}^1_R(M'', N) \to \ldots$$

Proof Similar to the foregoing. □

7.1.10 Theorem

1. If P is projective then $\mathrm{Ext}^n_R(P, N) = 0$ for all N and all $n > 0$.

2. If E is injective then $\mathrm{Ext}^n_R(M, E) = 0$ for all M, all $n > 0$.

Proof

1. The $\mathrm{Ext}^n_R(P,N)$ are independent of the chosen resulotion of P, so choose : $\ldots \to 0 \to 0 \to P_0 \underset{\pi}{\longrightarrow} P \to 0$ with $P_0 = P$ and $\pi = I_P$. The claim follows easily.

2. Similar to 1. □

7.1.11 Corollary

If $\mathrm{Ext}^1_R(M,X) = 0$ for all X then M is projective. If $\mathrm{Ext}^1_R(X,N) = 0$ for all X then N is injective.

We finish by mentioning the following results.

7.1.12 Theorem

1. For all $n \in \mathbb{Z}, \mathrm{Ext}^n_R(\sum_k M_k, N) \cong \prod_k \mathrm{Ext}^n_R(M_k,N)$.

2. For all $n \in \mathbb{Z}, \mathrm{Ext}^n_R(M, \prod_k N_k) \cong \prod_k(\mathrm{Ext}^n_R(M,N_k)$.

7.1.13 Remark

For a finite direct sum we obtain

$$\mathrm{Ext}^n_R(\oplus_k M_k, N) \cong \oplus_k \mathrm{Ext}^n_R(M_k,N)$$

7.1.14 Proposition

Let r be central in R, $\mu_r : M \to M$ is the left multiplication by r (this is R-linear now !), then $\mu_n : \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N)$, is also multiplication by r (a similar statement holds with respect to the second variable.

Proof We may start from the following diagram with identical rows :

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] \arrow[d, "\mu_r"] & 0 \\
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We defined $\mu_n : \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N)$ by taking a chain map over μ, say g, $(g_n : P_n \to P_n)$ then apply the functor $\mathrm{Hom}_R(-,N)$ to the diagram and define μ_n (z_n + boundaries) = $g_n z_n$ + boundaries. We established that any choice of a chain map over μ defines the same μ_n, in particular we may take g by putting $g_n : P_n \to P_n$ equal to multiplication by r, this g is a chain map over μ and μ_n (z_n+boundaries)= rz_n+boundaries = r (z_n+boundaries). □

The foregoing material suffices for the understanding of next sections. Full detail about elementary homological algebra are contained in [20] or [19].

[19] D. G. Northcott, *An introduction to Homological Algebra*, Cambridge Univ. Press. London, 1960.

[20] Joseph J. Rotman, *Notes on Homological Algebra*, Van Nostrand, Math. Studies, 26, New York, 1970.

[18] C. Năstăsescu, F. van Oystaeyen, *Dimensions of Ring Theory*, D. Reidel Publ. Co., Mathematics and its Applications, Dordrecht, Boston, 1987.

7.2 Projective Dimension

7.2.1 Definition

Left A-modules M and N are **projectively equivalent** if $M \oplus P = N \oplus P$ for some projective left A-module P. This is clearly an equivalence relation, indeed if $M_1 \oplus P = M_2 \oplus P$ and $M_2 \oplus Q = M_3 \oplus Q$ then $M_1 \oplus Q \oplus P = M_3 \oplus Q \oplus P$ and $Q \oplus P$ is projective as a direct sum of projective modules.

If $M \sim M_1$ (writing $\sim$ for projective equivalence) and $N \sim N_1$, then $M \oplus N \sim M_1 \oplus N_1$. Write [M] for the projective equivalence class of M. Then [P] with P a projective left A-module is a unit for the direct sum. Let $Q(A)$ be the class of left A-modules modulo projective equivalence then Q/A is an abelian monoid.

7.2.2 Lemma (Schanuel's lemma)

Consider the following exact sequences of left A-modules :

$$0 \longrightarrow M \longrightarrow P \xrightarrow[\pi]{} N \longrightarrow 0$$

$$0 \longrightarrow M' \longrightarrow P' \xrightarrow[\pi']{} N \longrightarrow 0$$

When P and P' are projective then $M' \oplus P = M \oplus P'$.

Proof Consider a left A-submodule $L = \{(x, x') \in P \oplus P', \pi(x) = \pi'(x')\}$ in $P \oplus P'$ and observe that the map Π, $\Pi : L \to P$, $(x, x') \mapsto x$, is surjective. Indeed for $x \in P$ surjectivity of π' yields an $x' \in P'$ such that $\pi'(x') = \pi(x)$, then $(x, x') \in L$ and $\Pi(x, x') = x$. Projectivity of P yields the splitting of the sequence $0 \to \mathrm{Ker}(\Pi) \to L \to P \to 0$, hence $L = P \oplus \mathrm{Ker}(\Pi)$. Now $\mathrm{Ker}(\Pi) = \mathrm{Ker}(\pi') \cong M'$, hence $L = P \oplus M'$. A completely symmetric argument leads to $L = P' \oplus M$ (using $\pi_1 : L \to P', (x, x') \mapsto x'$), hence the statement follows. $\square$

In the situation of the lemma it then follows that $[M] = [M']$. We may therefore define a map $\mathcal{P}$, $\mathcal{P} : A\text{-mod} \to \mathcal{P}(A), N \mapsto [M]$, where there is an exact sequence

$$0 \to M \to P \to N \to 0$$

with P projective. One easily sees that $\mathcal{P}(N \oplus N') = \mathcal{P}(N) \oplus \mathcal{P}(N')$ and if $[N] = [N']$ then $N \oplus P = N' \oplus P'$ for projective left A-modules P and P', thus $\mathcal{P}(N) + \mathcal{P}(P) = \mathcal{P}(N') + \mathcal{P}(P')$ where

$\mathcal{P}(P)$ and $\mathcal{P}(P')$ are $[0]$, hence $\mathcal{P}(N) = \mathcal{P}(N')$; so we arrive at a well defined endomorphism $\mathcal{P} : \mathcal{P}(A) \to \mathcal{P}(A)$ which we will call the **projective shift** of A.

For $M \in A$-mod we define the **projective dimension** $\mathrm{pd}(M)$ on the minimal integer or such that there exists a projective resolution of M :

$$(*) \qquad 0 \to P_n \to \ldots \to P_1 \to P_1 \to P_0 \to M \to 0$$

this is an exact sequence with all P_i projective.

7.2.3 Lemma

For $M \in A$-mod, $pd(M) = \min\{n \in \mathbb{N}, \mathcal{P}^n(M) = 0\}$.

Proof Let $(d) : 0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ be a projective resolution where $d_n : P_n \to P_{n-1}$. Then $\mathcal{P}^n([M]) = [\mathrm{Ker}(d_{n-1})]$ (this is trivial for $n = 1$ and then follows easily by induction). Thus $\mathcal{P}^n([M]) = [0]$ if and only if $\mathrm{Ker}(d_{n-1})$ is projective, meaning that the resolution stops at P_n. $\square$

For $N \in A$-mod we have a functor $\mathrm{Hom}_A(-, N)$ from A-mod to $\mathbb{Z}$-mod (abelian groups), we let $\mathrm{Ext}_A(-, N)$ be the cohomology associated to the functor
$\mathrm{Hom}_A(-, N)$. The **injective dimension** of $M \in A$-mod (notation: $id(M)$) is defined dually to the projective dimension, i.e. it is the minimal integer for which there exists an injective resolution of M:

$$0 \to M \to I^0 \to I^1 \to \ldots \to I^n \to 0$$

where the I^j are injective left A-modules. The duality actually leads to the equality of injective and projective dimension, this is the result of the following theorem.

7.2.4 Theorem (The Global Dimension Theorem)

For a ring A the following numbers are equal.

1. $n_1 = \sup\{\mathrm{pd}(M), M \in A{-}\mathrm{mod}\}$
2. $n_2 = \sup\{\mathrm{id}(M), M \in A{-}\mathrm{mod}\}$
3. $n_3 = \sup\{\mathrm{d}, \mathrm{Ext}^d_A(M, N) \neq 0; M, N \in A{-}\mathrm{mod}\}$

Proof If n_1 is finite then every left A-module has a projective resolution of length at most n_1. For $k > n_1$, applying $\mathrm{Ext}_A(-, N)$ to any projective resolution of M (having length at most n_1) yields $\mathrm{Ext}^k_A(M, N) = 0$, hence $n_3 \leq n_1$. Similarly $\mathrm{Ext}_A(M, N)$ can also be defined as the cohomology induced by the functor $\mathrm{Hom}_A(M, -)$, then appying it to any choice of injective resolution of M, yields $n_3 \leq n_2$.

If n_2 is finite then we may construct for any $M \in A$-mod a resolution of length n_3 where every component except perhaps the last one is projective. By dimension shift : $0 = \mathrm{Ext}^{n_3+1}(M, N) = \mathrm{Ext}^1_A(P_{n_3}, N)$ for all N. This implies that P_{n_3} is also projective and we have a projective resolution of length n_3, Hence $\mathrm{pd}(M) \leq n_3$ for every M, or $n_1 \leq n_3$. A dual argument leads to $n_2 \leq n_3$. $\square$

7.2.5 Definition

The number defined in Theorem 7.2.4. is called the **left global dimension** of A, it will be denoted by : $\mathrm{l.gl.dim}(A)$.

We may facilitate the calculation of the left global dimension somewhat by restricting the family of A-modules of which the projective dimension has to be calculated, to the class of cyclic left A-modules.

7.2.6 Lemma

$\mathrm{l.gl.dim}(A) = \sup\{\mathrm{pd}(A/I), I \text{ a left ideal of } A\}$.

Proof Recall Baer's criterion asserting that a ring is semisimple if and only if every left ideal is injective.

A left ideal I yields an exact sequence in A-mod :

$$0 \to I \to A \to A/I \to 0$$

leading to a sequence of $\mathbb{Z}$-modules :

$$\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(I, M) \to \mathrm{Ext}^1_A(A/I, M) \to 0$$

By Baer's criterion M is injective if and only if $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(I, M)$ is surjective, equivalently if $\mathrm{Ext}^1_A(A/I, M) = 0$ for all left ideals I of A. Obviously $d = \sup\{\mathrm{pd}(A/I), I$ a left ideal of $A\} \leq \mathrm{l.gldim}(A/I)$. Assume the foregoing inequality to be a strict one. Then we can select $M \in A$-mod with a resolution

$$0 \to M \to E^0 \to E^1 \to \ldots \to E^{d-1} \to N \to 0$$

where E^j is injective and N not being injective. By dimension shift :

$$0 = \mathrm{Ext}^{d+1}_A(A/I, M) \cong \mathrm{Ext}^1_A(A/I, N)$$

for each left ideal I of A. Therefore N is injective, a contradiction. □

7.2.7 Proposition

We have $\mathrm{l.gldim}(A) = 0$ if and only if A is semisimple.

Proof Every left A-module has a resolution of length zero if every left A-module is projective, i.e. if and only if every short exact sequence is split. Hence it follows that every submodule of an $M \in A$-mod is a direct summand and has a complement, therefore A is semisimple. □

7.2.8 Definition

A ring A is **left hereditary** if and only if left submodules of left projective modules are again projective.

7.2.9 Proposition

The following statements are equivalent :

1. A is left hereditary.
2. Every left ideal of A is projective in A-mod.

Proof

1. $\Rightarrow$ 2. A is projective as a left module over itself.
2. $\Rightarrow$ 1. It suffices to show that every submodule of a free left A-module is projective since projective left A-modules are direct summands of free left A-modules. Consider a submodule M of a free left A-module F, say F has basis $\{x_i, i \in \mathcal{J}\}$ and consider a well-ordering on $\mathcal{J}$. For $i \in J$ look at $F_i = \oplus_{j<i} Ax_j$ and define a map $\pi_i : Ax_i + F_i \to A, ax_i + f \mapsto a$. Put $M_i = M \cap F_i$ and put $\varphi_i = \pi_i | M_i$. Then $\mathrm{Ker}(\varphi_{i+1}) = M_i$. Put $L_i = \varphi_{i+1}(M_{i+1})$, then we obtain an exact sequence :
$$0 \to M_i \to M_{i+1} \to L_i \to 0$$
As a left ideal L_i is projective hence the sequence above is split and we obtain the recurrence relation $M_{i+1} = L_i \oplus M_i$; so by inductively continuing we obtain that $M_\alpha = \oplus_{\beta<\alpha} L_\beta$ for any ordinal. Therefore the result follows by taking for α the ordinal of J because every submodule M of a free left A-module is now a direct sum of left ideals of A hence projective. $\square$

7.2.10 Corollary

A ring A has $\mathrm{l.gldim}(A) = 1$ if and only if A is left hereditary.

Proof For a left ideal I of A consider the exact sequence :
$$0 \to I \to A \to A/I \to 0$$
When A is left hereditary, the foregoing squence is a projective resolution of A/I, thus $\mathrm{Ext}^k_A(A/I, M) = 0$ if $k > 1$ for all left ideals I and left A-modules M. Conversely, look at a projective $P \in A$-mod and let $N \subset P$ be a left A-submodule; then the sequence :
$$0 \to N \to P \to P/N \to 0$$
is a resolution of P/N, then $\mathrm{pd}(P/N) = 1$ yields $[N] = \mathcal{P}([P/N]) = 0$, hence N is projective. $\square$

By left-right symmetry we may define the right gl-dimension : $\mathrm{rgldim}(A)$ and derive foregoing results by changing to right A-modules everywhere. However the left and right global dimensions do not agree as is established by L. Small for the following example.

Look at the ring $A = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $\mathrm{lgldim}A = 1$ and right dimension $\mathrm{rgldim}A > 1$. First observe that $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & n\,\mathbb{Z} \end{pmatrix}$ is a strictly descending chain of right ideals, so A is not Artinian, hence not semisimple, thus $\mathrm{lgldim}(A) > 0$. We have to check that A is left but not right hereditary. Consider a left ideal L of A and $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in L$; then we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} = e_{11} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + e_{22} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

which establishes that $L = e_{11}L \oplus e_{22}L$, where e_{11}, e_{22} are the matrix units. Now $e_{11}L$ is a left $\mathbb{Q}$-submodule of $\mathbb{Q} \oplus \mathbb{Q}$, $e_{22}L$ is a left $\mathbb{Z}$-submodule of $\mathbb{Z}$, hence both components are projective and therefore so is L. We have checked that A is left hereditary. Now the right ideal Ae_{12} has multiplication $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & xc \\ 0 & 0 \end{pmatrix}$ hence it is $\mathbb{Q}$ as a $\mathbb{Z}$-module but we know that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module (every $\mathbb{Z}$-linear $\mathbb{Q} \to \mathbb{Z}$ is the zero map), therefore A is not right hereditary.

Now using the flat version of projective dimension and the corresponding weak global dimension of a ring one can actually prove the following.

7.2.11 Theorem

If A is Noetherian, then the left global dimension of A equals the right global dimension.

We end this section by the "change of ring" theorem, this is very useful for the calculation of the global diension of the Rees ring of several nice filtered rings and the Weyl algebra (see 7.4.).

We start by some lemmas concerning projective dimension.

7.2.12 Lemma

Let $M_i, i \in J$, be a family of left A-modules,

$$\mathrm{pd}\left(\oplus_{i \in J} M_i\right) = \sup\{\mathrm{pd}(M_i), i \in J\}$$

Proof If $[M] + [N] = [0]$ in $\mathcal{P}(A)$ then $M \oplus N$ is projective, but then M as well as N is projective, hence $[M] = [N] = [0]$. Now we have $\mathcal{P}^n(\oplus_{i \in J} M_i) = \sum_{i \in J} \mathcal{P}^n(M_i)$, by (an iteration of) the above we obtain that $\mathcal{P}^n(\oplus_{i \in J} M_i)$ is zero only if each $\mathcal{P}^n(M_i) = 0$ for $i \in J$, hence the result follows. $\square$

7.2.13 Lemma

Consider the following exact sequence in A-mod

$$0 \to M' \to M \to M'' \to 0$$

1. $\mathrm{pd}(M') \leq \max\{\mathrm{pd}(M), \mathrm{pd}(M'')\}$
2. $\mathrm{pd}(M) \leq \max\{\mathrm{pd}(M') + 1, \mathrm{pd}(M'')\}$
3. $\mathrm{pd}(M'') \leq \max\{\mathrm{pd}(M) + 1, \mathrm{pd}(M') + 1\}$

Moreover, if $\mathrm{pd}(M') = 1$ and $\mathrm{pd}(M'') > 1$ then $\mathrm{pd}(M'') = 1 + \mathrm{pd}(M)$.

Proof If M'' is projective then the sequence is split and thus $M = M' \oplus M''$, $[M] = [M']$ and then $\mathrm{pd}(M) = \mathrm{pd}(M')$. If M is projective, then $\mathcal{P}([M'']) = [M']$ and then $\mathrm{pd}(M'') \leq \mathrm{pd}(M')+1$. So we are reduced to considering the case where neither M'' nor M is projective. Consider a surjective morphism $\pi : P \to M$, where P is projective, so $0 \to \mathrm{Ker}(\pi) = K \to P \to M \to 0$ is exact. We have the following exact sequences in A-mod :

a. $0 \to K \to P \to M \to 0$

b. $0 \to \pi^{-1}(M') \to P \to M'' \to 0$

c. $0 \to K \to \pi^{-1}(M') \to M' \to 0$

From the projective shift characterization of projective dimension we obtain that $\mathrm{pd}(M)$ and $\mathrm{pd}(M'')$ are both nonzero, $\mathrm{pd}(K) = \mathrm{pd}(M) - 1, \mathrm{pd}(\pi^{-1}(M')) = \mathrm{pd}(M'') - 1$. By induction the second inequality implies

$$\mathrm{pd}(\pi^{-1}(M')) \leq \max\{\mathrm{pd}(K) + 1, \mathrm{pd}(M')\}$$

yielding : $\mathrm{pd}(M'') \leq \max\{\mathrm{pd}(M + 1), \mathrm{pd}(M') + 1\}$. This proves inequality 3. The other inequalities are established in a similar way. The last statement is clear. □

7.2.14 Theorem (First change of rings theorem)

Let A be a ring and x a central regular element of A. If M is an A/Ax-module with $\mathrm{pd}_{A/Ax}(M) < \infty$ then we have : $\mathrm{pd}_A(M) = 1 + \mathrm{pd}_{A/Ax}(M)$.

Proof We argue by induction on $\mathrm{pd}_{A/Ax}(M)$. In case $\mathrm{pd}_{A/Ax}(M) = 0$ then M is a projective left A/Ax-module and therefore a direct summand of some free A/Ax-module, N say. Regularity of x yields that $0 \to Ax \to A \to A/Ax \to 0$ is a free resolution of A/Ax, hence $\mathrm{pd}_A(A/Ax) \leq 1$. The foregoing sequence is not split since x is a non-zerodivisor. Thus A/Ax is not projective in A-mod, i.e. $\mathrm{pd}_A(A/Ax) = 1$. Also we have $\mathrm{pd}_A(N) = \mathrm{pd}_A((A/Ax)^{\oplus w}) = 1$ and $\mathrm{pd}(M) \leq 1$. Since M is an A/Ax-(left)module, we have $xM = 0$. A projective module being a submodule of a free left module it is in particular a faithful module; therefore M is not projective, thus $\mathrm{pd}_A(M) = 1$, proving the case $n = 0$. Now, if $\mathrm{pd}_{A/Ax}(M) > 0$, then look at the exact sequence :

$$0 \to K \to F \to M \to 0$$

for some free A/Ax-module F, where $K \neq 0$, since M is not A/Ax-projective. From $\mathcal{P}([M]) = [K]$ we obtain : $\mathrm{pd}_{A/Ax}(K) = \mathrm{pd}_{A/Ax}(M) - 1$, thus $\mathrm{pd}_A(K) = \mathrm{pd}_{A/Ax}(K) + 1 = \mathrm{pd}_{A/Ax}(M)$ by induction. The foregoing Lemma implies : $\mathrm{pd}_A(M) \leq \mathrm{pd}_A(K) + 1 = \mathrm{pd}_{A/Ax}(M) + 1$, with equality when $\mathrm{pd}_{A/Ax}(M) > 1$.

Therefore, we reduce the proof to the case $\mathrm{pd}_{A/Ax}(M) = 1$, then we have to establish that $\mathrm{pd}_A(M) = 2$. But by the inequality proved before it suffices to show that $\mathrm{pd}_A(M) > 1$. So write $M = P/Q$ for some projective left A-module P. Look at the exact sequences :

$$0 \to Q/Px \to P/Px \to M \to 0$$

$$0 \to Px/Qx \to Q/Qx \to Q/Px \to 0$$

where $Px \subset Q$ because $x(P/Q) = 0$. Since P is projective in A-mod, P/Px is projective in A/Ax-mod and consequently Q/Px is also a projective left A/Ax-module since $\mathcal{P}([M]) = 0$. Hence the second exact sequence above is split, hence $M \cong Px/Qx$ is a direct summand of Q/Qx. Consequenly Q/Qx cannot be projective, otherwise M would be too, then Q is not projective in A-mod, and M is not a quotient of a projective by a projective, or $\mathrm{pd}_A(M) > 1$.□

7.2.15 Corollary

$\mathrm{gldim}K[[X_1, \ldots, X_n]] = n$.

Proof First show $\mathrm{gldim}K[[X]] = 1$. Applying the theorem to the maximal ideal (X) of $K[[X]]$, the followiing statement follows : for every K-vectorspace M, $\mathrm{pd}_{K[[X]]}(M) = 1$, hence $\mathrm{gldim}K[[X]] \leq 1$. Since $K[[X]]$ is not a direct product of fields it is not semisimple, thus $\mathrm{gldim}K[[X]] = 1$. Now $K[[X]][[Y]] = K[[X, Y]]$, the foregoing argument may be repeated over $K[[X]]$, and so on, till we reach $K[[X_1, \ldots, X_n]]$. □

7.2.16 Corollary

Let x be a regular central element in A, then $\mathrm{gldim}(A/Ax) = n$ entails $n + 1 \leq \mathrm{gldim}(A)$.

7.2.17 Theorem (Second change of rings theorem)

Let x be a central regular element in a ring A and let M be a left A-module such that x is regular on M (i.e. M is x-torsion free), then : $\mathrm{pd}_{A/Ax}(M/xM) \leq \mathrm{pd}_A(M)$.

Proof The proof is also by induction. If $\mathrm{pd}_A(M) = 0$ then $\mathrm{pd}_{A/Ax}(M/xM) = 0$ is easily checked. If $\mathrm{pd}_A(M) > 0$ then consider $0 \to K \to F \to M \to 0$, where F is a free left A-module. By induction we have $\mathrm{pd}_{A/Ax}(K/Kx) \leq \mathrm{pd}_A(K)$. Taking the tensor product $\otimes_A A/Ax$ leads to the sequence :

$$0 \to \mathrm{Tor}_1^A(M, A/Ax) \to K/Kx \to F/Fx \to M/Mx \to 0$$

Since $\mathrm{Tor}_1^A(M, A/Ax) = \mathrm{Ann}(x) = 0$ and F/xF is a projective A/Ax-module we have :
$\mathrm{pd}_{A/Ax}(M/xM) = 1 + \mathrm{pd}_{A/Ax}(K/xK) \leq \mathrm{pd}_A(M)$ □

7.2.18 Lemma

$\mathrm{pd}_{A[X]}(A[X] \otimes_A M) = \mathrm{pd}_A(M)$.

Proof By the second change of rings theorem :

$$\mathrm{pd}_A(M) \leq \mathrm{pd}_{A[X]}(A[X] \otimes_A M)$$

The other inequality follows from looking at a projective resolution of M in A-mod and by tensoring with $A[X]$ we find a projective resolution of $A[X] \otimes_A M$ in $A[X]$-mod. Thus $\mathrm{pd}_{A[X]}(A[X] \otimes_A M) \leq \mathrm{pd}_A(M)$. □

7.2.19 Theorem (Hilbert's syzygy theorem)

We have : $\text{gldim}A[X_1, \ldots, X_n] = \text{gldim}(A) + n$.
If K is a field : $\text{gl.dim}K[X_1, \ldots, X_n] = n$ (this is the original Hilbert theorem).

Proof It suffices to establish the claims for $n = 1$, the general case follows by repetition. By the first change of rings theorem : $1 + \text{gldim}(A) \leq \text{gldim}A[X]$.

Conversely, for an $A[X]$-module M consider the sequence :

$$(*) \qquad 0 \to A[X] \otimes_A M \xrightarrow[f]{} A[X] \otimes_A M \xrightarrow[\mu]{} M \to 0$$

where μ is the multiplication given by $\mu(a \otimes m) = am$ and f is defined by $f(b \otimes m) = b(X \otimes m - 1 \otimes Xm)$. The sequence (*) is short exact, indeed, every element of $A[X] \otimes_A M$ can be written as $c = X^k \oplus m_k + \ldots + X \otimes m_1 + 1 \otimes m_0$ for some $m_0, \ldots, m_k \in M$, the leading term of $f(c)$ is $X^{k+1} \otimes m_k$, hence f is injective. Moreover : $(\mu \circ f)(X^j \otimes m_j) = \mu(X^j(X \otimes m_i - 1 \otimes Xm_i)) = X^{i+1}m_i - X^{i+1}m_i = 0$. To show that $\text{Ker}\mu \subset \text{Im}f$ we use induction on k. If $k = 0$ then $1 \otimes m \in \text{Ker}\mu$ if and only if $m = 0$. If $k > 0$ then for $b \in \text{Ker}\mu$ consider $b' = b - f(X^{k-1} \otimes m_k)$. From $\mu(f(X^{k-1} \otimes m_k) = 0$ we have $\mu(b) = \mu(b')$. On the other hand, this polynomial expression has leading term of degree $k - 1$, so by induction $b' = f(b'')$ for some $b'' \in A[X] \otimes_A M$ and it follows that $b = f(b'' + X^{k-1} \otimes m_k)$. One checks easily that μ is also surjective. Now we obtain :

$$\text{pd}_{A[X]}(M) \leq 1 + \text{pd}_{A[X]}(A[X] \otimes_A M) = 1 + \text{pd}_A(M) \leq 1 + \text{gldim}A$$

hence $\text{gldim}A[X] \leq 1 + \text{gldim}A$, leading to the desired equality. The final statement follows from the foregoing plus the fact that $\text{gldim}(K) = 0$. $\square$

7.3 Projective and Global Dimension for Filtered Rings

The category R-filt of filtered left R-modules over the filtered ring R with filtration FR is not a Grothendieck category, yet it allows enough freedom to develop some homological algebra in it. An $L \in R$-filt is said to be **filt-free** if it is a free left R-module having a basis $\{e_i, i \in J\}$ such that there is a family $\{d_i, i \in J\}$ in $\mathbb{Z}$ such that : $F_nL = \sum_{i \in J} F_{n-d_i}Re_i, n \in \mathbb{Z}$, and $e_i \notin F_{d_i-1}R, i \in J$. Some $P \in R$-filt is said to be **filt-projective (projective in R-filt)** if it is a direct summand of a filt-free L in R-filt, i.e. $L = P \oplus P'$ and $F_nL = F_nP \oplus F_nP'$ for every $n \in \mathbb{Z}$.

7.3.1 Lemma

Consider a filtered R-module L with filtration FL.

1. If L is filt-free with basis $\{e_i, i \in J\}$, then $G(L)$ is gr-free $G(R)$-module with homogeneous basis $\{\sigma(e_i), i \in J\}$.

2. If $G(L)$ is gr-free with homogeneous basis $\{\overline{e}_i = \sigma(e_i), i \in J\}$, where $e_i \in F_{d_i}L - F_{d_i-1}L$, then if FL is discrete L is filt-free with basis $\{e_i, i \in J\}$.

3. If $\overline{M} \in G(R)$-gr is gr-free, then there exists a filt-free L with filtration FL such that $G(L) = \overline{M}$ in $G(R)$-gr.

4. If L is a filt-free with basis $\{e_i, i \in J\}$ and $f : \{e_i, i \in J\} \to M$ is a map to $M \in R$-filt such that $f(e_i) \in F_{s+d_i}M$ for some $s \in \mathbb{Z}, i \in J$, then there is a unique filtered morphism $g : L \to M$ extending the map f to L.

5. If L is filt-free, $M \in R$-filt, are such that there is a graded morphism of degree s, $g : G(L) \to G(M)$ then there is a filtered morphism of degree s, $f : L \to M$, such that $G(f) = g$.

6. Let L be filt-free with basis $\{e_i, i \in J\}$, then FL is separated if and only if FR is separated. If FR is discrete and $\{d_i, i \in J\}$ is bounded below then FL is discrete. In case J is finite and FR is a complete filtration, then FL is complete.

Proof Easy. Let us just prove 5. Put $L = \oplus_{i\in J} Re_i$, $G(L) = \oplus_{i\in J} G(R)\sigma(e_i)$ and $g(\sigma(e_i)) = \overline{\xi}_i \in G(M)_{d_i+s}$ for all $i \in J$. Define $f : L \to M$ by taking $f(e_i) = \xi_i$ where $\xi_i \in F_{d_i+s}M - F_{d_i+s-1}M$ represents $\overline{\xi}_i$; it is easily checked that f is a filtered morphism of degree s. □

7.3.2 Corollary

1. $M \in R$-filt has good filtration FM if and only if there is a filt-free L of finite rank such that there is a strict epimorphism $L \longrightarrow\!\!\!\!\rightarrow M$.

2. If FM is a good filtration on M and R is complete and FM is separated, then M is complete for FM. If L is filt-free with basis $\{e_i, i \in J\}$, then $\widetilde{e}_i \in \widetilde{L}_{d_0}$ (i.e. $\widetilde{e}_i = e_i T^{d_i}$ where T is the central regular homogeneous element of $\widetilde{R}$), for $i \in J$, forms a homogeneous $\widetilde{R}$-basis for $\widetilde{L}$. A filt-projective module P is a filt-direct summand of a filt-free L, hence $\widetilde{L} = \widetilde{P} \oplus \widetilde{P'}$ follows; this allows to state the following

7.3.3 Lemma

1. $M \in R$-filt is filt-free if and only if $\widetilde{M}$ is gr-free in $\widetilde{R}$-gr.

2. $P \in R$-filt is filt-projective if and only if $\widetilde{P}$ is gr-projective in $\widetilde{R}$-gr.

3. Assume now that R is complete with respect to FR, then $L \in R$-filt with separated filtration FL is filt-free of finite rank if and only if $G(L)$ is a gr-free $G(R)$-module of finite rank.

Proof 1. and 2. are easy enough.

3. If L is filt-free of finite rank then it follows from the foregoing results that $G(L)$ is gr-free of finite rank in $G(R)$-gr. Conversely, let $G(L)$ be gr-free of finite rank, say $G(L) = \oplus_{i=1}^{s} G(R)\sigma(e_i)$, where $e_i \in F_{d_i}L - F_{d_i-1}L$ and $\sigma(e_i) = e_i + F_{d_i-1}L$ (FL is separated !). As in the proof of Proposition 4.2.10, we then obtain : $F_nL = \sum_{i=1}^{s} F_{n-d_i}Re_i$, for all

$n \in \mathbb{Z}$, hence $L = \sum_{i=1}^{s} Re_i$. Since FR is separated it is clear that $e_1, \ldots, e_s$ are R-linear independent, hence L is filt-free of rank s. $\square$

Given a diagram of filtered morphisms :

$$(p) \qquad \begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "\pi"'] & M' \arrow[r] & 0 \end{tikzcd}$$

where π is a strict filtered epimorphism. If P is filt-projective, then we may find a filtered morphism $h : P \to M$, such that $\pi \circ h = g$. Indeed the strictness of π yields that $\widetilde{\pi}$ is surjective and the projectivity of $\widetilde{P}$ in $\widetilde{R}$-gr leads to a graded morphism $\widetilde{h} : \widetilde{P} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \widetilde{h} = \widetilde{g}$, then we obtain h by dehomogenization of $\widetilde{h}$. Hence we arrive at the following.

7.3.4 Proposition

For any $P \in R$-filt :

1. If P is filt-projective, then $G(P)$ is projective in $G(R)$-gr.

2. P is filt-projective if and only if for every strict exact sequence in R-filt : $M \longrightarrow^{\pi} M' \longrightarrow 0$, and for any filtered morphism $g : P \to M'$, there exists a filtered morphism $h : P \to M$, such that the diagram (p) is commutative.

Proof

1. Follows from $G(P) \cong \widetilde{P}/T\widetilde{P}$ and Lemma 7.3.3.

2. Follows from the remarks preceding the Proposition. $\square$

For exhaustively filtered R-modules M and N we have the exhaustively filtered $\mathbb{Z}$-module $\mathrm{HOM}_{FR}(M, N)$.

7.3.5 Proposition

If FM is good and $N \in R$-filt then we have $\mathrm{HOM}_{FR}(M, N) = \mathrm{Hom}_R(M, N)$.

Proof Let $m_1, \ldots, m_s$ be R-generators of M such that for all $n \in \mathbb{Z}$, $F_n M = \sum_{i=1}^{s} F_{n-d_i} R m_i$. Take an R-linear map $f : M \to N$. Choose $t \in \mathbb{Z}$ such that $f(m_i) \in F_{d_i+t} N$ for all $i \in \{1, \ldots, s\}$. Then $f \in F_t \mathrm{HOM}_R(M, N)$, hence $\mathrm{Hom}_R(M, N) = \mathrm{HOM}_{FR}(M, N)$. $\square$
We mention the following without proof.

7.3.6 Proposition

For M and N in R-filt :

1. If FN is separated, then $F\mathrm{HOM}_{FR}(M,N)$ is separated.
2. If M is finitely generated and FN is discrete, then $F\mathrm{HOM}_{FR}(M,N)$ is discrete.
3. If FN is complete, then $F\mathrm{HOM}_{FR}(M,N)$ is complete.

   For $M, N \in R$-filt there is a canonical map :

   $$\Psi(= \Psi(M,N)) : G(\mathrm{HOM}_{FR}(M,N)) \to \mathrm{HOM}_{G(R)}(G(M), G(N)),$$
   $$f \in F_p\mathrm{HOM}_{FR}(M,N) \mapsto \Psi(f_{(P)})$$

   where $\Psi(f_{(P)})$ is given as follows : $\Psi(f_{(P)})(x_g) = f(x)_{p+g}$. The nice situation arises when Ψ is an isomorphism.

7.3.7 Lemma

For every $M, N \in R$-filt, $\Psi(M,N)$ is a graded monomorphism. If M is filt-projective, then $\Psi(M,N)$ is an isomorphism.

Proof That Ψ is a graded morphism is clear by definition. If $\Psi(f_{(P)}) = 0$ for some $f \in F_P\mathrm{HOM}_{FR}(M,N)$ then $f(x)_{p+g} = 0$ for every $x \in F_qM$. Thus $f(F_qM) \subset F_{p+q-1}N$ for all $q \in \mathbb{Z}$, hence we have :$f \in F_{p-1}\mathrm{HOM}_{FR}(M,N)$. Consequently : $f_{(P)} = 0$. If M is filt-projective, then $L = M \oplus Q$ in R-filt for some filt-free L. Since HOM_{FR} commutes with direct sums, we only have to establsh the second statement in case M is filt-free. So, let $\{m_i, i \in J\}$ be a basis for the filt-free M. Then $\{m_{i(d_i)}, i \in J\}$ is a homogeneous $G(R)$-basis for $G(M)$. Giving $g \in \mathrm{HOM}_{G(R)}(G(M), G(N))$, then $g(m_{i(d_i)}) = x_{p+d_i}$ for $i \in J$. Define $f : M \to N$ by putting $f(m_i) = x^i$ and check that $f \in F_P\mathrm{HOM}_{FR}(M,N)$, moreover $\Psi(M,N)(f) = g$. Consequenlty $\Psi(M,N)$ is an isomorphsim. □

Call a ring R **left regular** if every finitely generated left R-module has finite projective dimension, equivalently every cyclic left R-module has finite projective dimension. For any group graded ring **left gr-regularity** is defined similarly in terms of objects of R-gr. We first observe a converse for Proposition 7.3.4(1).

7.3.8 Proposition

Assume FR is complete and $P \in R$-filt has separated filtration FP such that $G(P)$ is finitely generated in $G(R)$. If $G(R)$ is projective in $G(R)$-gr, then P is filt-projective.

Proof Since $G(P)$ is finitely generated it follows from Proposition 4.3.10 that FP is good, hence there is a strict exact sequence in R-filt :

$$0 \to K \to L \to P \to 0$$

with L being filt-free of finite rank. By Lemma 7.3.2., $\widetilde{L}$ is a gr-free left $\widetilde{R}$-module of finite rank and $\widetilde{L}/T\widetilde{L} = G(L)$ is a gr-free $G(R)$-module of finite rank. Then $G(L) = G(P) \oplus G(K)$ follows from the exact sequence in $G(R)$-gr : $0 \to G(K) \to G(L) \to G(P) \to 0$. Put $N = P \oplus K$ in R-filt with FN defined by $F_nN = F_nP \oplus F_nK$, for all $n \in \mathbb{Z}$. Then $\widetilde{N} = \widetilde{P} \oplus \widetilde{K}$ and $\widetilde{N}/T\widetilde{N} = \widetilde{P}/T\widetilde{P} \oplus \widetilde{K}/T\widetilde{K} = \widetilde{L}/T\widetilde{L}$. Now FT is separated and $G(T)$ is gr-free of finite rank. By Lemma 7.3.3.(3), T is filt-free of finite rank, therefore P is filt-projective. □

7.3.9 Corollary

Let R be complete and $G(R)$ left Noetherian. If $G(R)$ is left gr-regular, then R is left regular and we have : $\mathrm{l.gldim}R \leq \mathrm{gr.l.gldim}G(R)$.

Proof Suppose $\mathrm{gr.lgldim}G(R) = n < \infty$, otherwise the statement is clear. Consider an arbitrary finitely generated R-module and put a good filtration FM on it. Hence we have a strict exact sequence in R-filt, $0 \to K \to L \to M \to 0$, where L is filt-free of finite rank. Since $G(R)$ is left Noetherian, $G(K) \subset G(L)$ is finitely generated. Since FR is separated we have that F is separated and thus FK is separated too. By Proposition 4.3.10, FK is good. Repeating this procedure for K, and so on, we arrive at a strict exact

$$0 \to K_n \to L_{n-1} \to \ldots \to L_0 \to M \to 0$$

where all L_i are filt-free of finite rank and K_n has separated filtration FK_n which is also a good filtration. From Theorem 3.2.21 we retain that :

$$0 \to G(K_n) \to G(L_{n-1}) \to \ldots \to G(L_0) \to G(M) \mapsto 0$$

is exact. Now all $G(L_i)$ are gr-free of finite rank and $G(K_n)$ is finitely generated. Since $n = \mathrm{gr.lgdim}G(R), G(K_n)$ is projective, hence by the foregoing proposition it follows that K_n is filt-projective, hence $\mathrm{pdim}_R M \leq \mathrm{pdim}G(M) \leq n$. Hence $\mathrm{lgldim}R \leq n$. From this proof it is also clear that R is left regular if $G(R)$ is left gr-regular. □

7.3.10 Lemma

Let R be a Noetherian ring, T a central regular element. Let A_T be the localization at the (central) Ore set $\{1, T, T^2, \ldots\}$. If M is a T-torsionfree A-module such that M/TM is a projective A/TA-module and $M_T = A_T \otimes M$ is a projective A_T-module, then M is a projective A-module (the converse is obvious).

Proof Let $\ldots \to L_1 \to L_0 \to M \to 0$ be a free resolution of M. Let $\mathbb{Z}[T]$ be the subring of A generated by 1 and T in $Z(A)$. For any A-module N there is an isomorphism of complexes of $\mathbb{Z}[T]_T$-modules :

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[T]_T \otimes \mathrm{Hom}_A(M,N) \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}[T]_T \otimes \mathrm{Hom}_A(L_0,N) \arrow[r] \arrow[d, "\cong"] & Z[T]_T \otimes \mathrm{Hom}_A(L_1,N) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{A_T}(M_T,N_T) \arrow[r] & \mathrm{Hom}_{A[T]}(L_{0,T},N_T) \arrow[r] & \mathrm{Hom}_{A_T}(L_{1,T},N_T)
\end{tikzcd}$$

So we have an isomorphism of $\mathbb{Z}[T]_T$-modules :

$$\mathbb{Z}[T]_T \otimes \mathrm{Ext}^i_A(M,N) \cong \mathrm{Ext}^i_{A_T}(M_T,N_T)$$

for each $i \geq 0$. Since M_T is projective we obtain for $i \geq 1$ that $\mathrm{Ext}_{A_T}(M_T,N_T) = 0$ and thus $\mathrm{Ext}^i_A(M,N)$ is T-torsion for all $i \geq 1$.

On the other hand, $\mathrm{pd}(M/TM) = 1$ by the first change of rings theorem. Since M is T-torsionfree we have an exact sequence in A-mod $0 \to M \underset{\mu_T}{\longrightarrow} M \to M/TM \to 0$, where μ_T is multiplication by T. So we arrive at the long exact sequence :

$$\ldots \to \mathrm{Ext}^2_A(M/TM,N) \to \mathrm{Ext}^2_A(M,N) \underset{\mu_T}{\longrightarrow} \mathrm{Ext}^2_A(M,N) \to$$
$$\to \mathrm{Ext}^3_A(M/TM,N) \to \ldots$$

Since $\mathrm{Ext}^i_A(M,N)$ is independent of the choice of resolution for M it follows that $\dot{\mu}_T$ is again multiplcation by T and we arrive at an isomorphism $\dot{\mu}_T : \mathrm{Ext}^2_A(M,N) \to \mathrm{Ext}^2_A(M,N)$. Combining this with the fact that $\mathrm{Ext}^2_A(M,N)$ is T-torsion, allows to conclude that $\mathrm{Ext}^2_A(M,N) = 0$ for any left A-module N. Hence $\mathrm{pd}_A(M) \leq 1$. From an exact sequence in A-mod :

$$0 \to K \to L \to N \to 0$$

where L is free and N is finitely generated, it follows that $\mathrm{Ext}^1_A(M,N) = 0$ if $\mathrm{Ext}^1_A(M,A) = 0$, therefore M is projective. That $\mathrm{Ext}^1_A(M,A) = 0$ follows from the exact sequence of right A-modules :

$$\mathrm{Ext}^1_A(M,A) \underset{\dot{\mu}_T}{\longrightarrow} \mathrm{Ext}^1_A(M,A) \to \mathrm{Ext}^2_A(M/TM,A) = 0$$

Surjectivity of $\dot{\mu}_T$ combined with the fact that $\mathrm{Ext}^1_A(M,A)$ if finitely generated as a right A-module but also a T-torsion module, yields that $\mathrm{Ext}^1_A(M,A) > 0$.

□

7.3.11 Theorem

Let A be a Noetherian ring, T a central regular element of A.

1. If A/TA and A_T are left regular, then A is left regular.
2. We have : $\mathrm{gldim}A \leq \max\{1 + \mathrm{gldim}(A/TA), \mathrm{gldim}A_T\}$

In case $\mathrm{gldim}(A/TA)$ is finite, then equality holds in the foregoing.

Proof

1. Will follow from the proof of 2.
2. If $\max\{1 + \mathrm{gldim}(A/TA), \mathrm{gldim}A_T\} = \infty$ then there is nothing to prove, so we assume this is a finite number n. If M is a finitely generated (left) A-module, let $t(M)$ be the T-torsion submodule of M : $0 \to t(M) \to M \to M/t(M) \to 0$ is exact in A-mod. Since A is Noetherian, $T^w t(M) = 0$ for some $w \in \mathbb{N}$. Applying induction on w and

the first change of rings theorem, we arrive at : $\mathrm{pd}_A(t(M)) \leq 1 + \mathrm{gldim}(A/TA) \leq n$. Now apply the foregoing Lemma to the finitely generated $M/t(M)$ and conclude that : $\mathrm{pd}_A(M/t(M)) \leq n$. This establishes the claim because $\mathrm{pd}_A(M) \leq n$ follows from the foregoing and exactness of

$$0 \to t(M) \to M \to M/t(M) \to 0$$

Note that also $\mathrm{gldim}A \geq 1 + \mathrm{gldim}(A/TA)$ when $\mathrm{gldim}(A/TA)$ is finite (again by the first change of rings theorem), since we always have $\mathrm{gldim}A \geq \mathrm{gldim}A_T$. Therefore the proof is complete. □

7.3.12 Corollary

Let R be a filtered ring such that $\widetilde{R} = A$ is Noetherian. Let $P \in R$-filt be projective in R-mod, then $P$is filt-projective if and only if $G(P)$ is projective in $G(R)$-mod.

Proof For $\widetilde{P}$ we have $\widetilde{P}/T\widetilde{P} \cong G(P)$ and $\widetilde{P}_T = \widetilde{A}_T \otimes \widetilde{P} \cong P[T, T^{-1}]$ and if P is projective, then $\widetilde{P}$ is projective over $\widetilde{R}_T = R[T, T^{-1}]$. Then P is filt projective if $G(P)$ is projective by the foregoing. □

It is possible to generalize 7.3.8. and 7.3.9. for Zariskian filtrations and even to some non-Zariskian filtrations using the last results obtained above. We refer to [13] for detail, e.g. Theorem 11 p. 68.

In the final section we look at an example where the global dimension of a filtered ring is actually smaller than the global dimension of the associated graded ring.

7.4 Global Dimension of the Weyl Algebras

We will present a calculation of the global dimension of the Weyl algebras involving a tensor product and therefore it is useful to use "flat dimension" in stead of projective dimension. A left module M over a ring R is flat if $- \otimes_R M$ is an exact functor. Projective modules are flat and in fact for a Noetherian ring R finitely generated flat modules are projective, so for the Weyl algebras the transition from projective to flat will not really change anything.

The flat dimension $\mathrm{fd}(M)$ of a module M is the minimal $n \in \mathbb{N}$ such that there exists a flat resolution of M : $0 \to F_n \to \ldots \to F_0 \to M \to 0$ where all F_i are flat modules.

7.4.1 Theorem

For a ring R the following numbers are equal

1. $\sup\{\mathrm{fd}(M), M \in R\text{-mod}\}$
2. $\sup\{\mathrm{fd}(M), M \in \text{mod-}R\}$, the right version of flat dimension being defined by left-right symmetry.
3. $\sup\{\mathrm{d} \in \mathbb{N}, \mathrm{Tor}^{\mathrm{R}}_{\mathrm{d}} \neq 0 \text{ for } M, N \text{ in } R\text{-mod}\}$.

Proof The proof is very similar to the case of projective dimension, we shall just provide an outline. Write n_1, n_2, n_3 for the numbers defined in $1, 2, 3$ resp. If n_2 is finite, then since any right R-module M has a flat resolution of length less than n_1, $\mathrm{Tor}_k^R(M, N) = 0$ for any R-module N if $n_2 < k$ (Tor may be calculated by flat resolutions). Hence $n_3 \leq n_1$. Assuming $n_3 < n_2$ means that there exists a right R-module M and a resolution of M :

$$0 \to K \to F_{n_3-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

where K is not flat and all F_i are flat. By dimension shifting :

$$0 = \mathrm{Tor}_{n_3+1}^R(M, N) = \mathrm{Tor}_1^R(K, N)$$

for all N_i therefore we arrive at flatness for K, a contradiction. So we established $n_2 = n_3$. The equality $n_1 = n_3$ follows in the same way, interchanging left and right modules.

7.4.2 Definition

The number defined in the foregoing theorem is called the **weak global dimension** of R, it is denoted by $\mathrm{w.gldim}(R)$. It is left-right symmetric!

7.4.3 Lemma

Let R be a Noetherian ring and G a divisible abelian group, then for M and N in R-mod with M a finitely generated R-module, we have :

$$\mathrm{Tor}_n^R(M, \mathrm{Hom}_{\mathbb{Z}}(M, G)) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}_R^n(M, N), G)$$

Proof
Define $\Psi : M \times \mathrm{Hom}_{\mathbb{Z}}(N, G) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M, N), G)$, by : $\Psi(m, f)(\theta) = f\theta(m)$. This map is R-bilinear and so it factorizes via the tensor product $M \otimes \mathrm{Hom}_{\mathbb{Z}}(N, G)$. If $M = R$, then this map is given by : $\Psi(1 \otimes f) = f\theta(1)$ and it is in fact an isomorphism. Indeed, any map in $\mathrm{Hom}_R(R, N)$ is of the form $.n$ for some $n \in N$, thus if $\nu \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(R, N), G)$ we consider the map $\mu : n \mapsto \theta(n) \in \mathrm{Hom}_{\mathbb{Z}}(N, G)$. Now mapping ν to $1 \otimes \mu$ we obtain the inverse map.

Since both tensor product and Hom commute with direct sums, the foregoing fact extends to finitely generated free R-modules and then to finitely generated projective modules (as direct summands of free modules). Now look at both sides of the equation as functors in M. Since G is divisible it is $\mathbb{Z}$-injective, thus the right hand side is covariant right exact. The left hand side is always covariant right exact. Applying both functors to a projective resolution we obtain a natural transformation :

$$\mathrm{Tor}_n^R(M, \mathrm{Hom}_{\mathbb{Z}}(N, G)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}_R^n(M, N), G)$$

Since R is Noetherian we may assume that the projective resolution chosen contain only finitely generated terms, thus the above map becomes an isomorphism.

$\square$

7.4.4 Theorem

For a Noetherian R, $\text{wdim}(M) = \text{pdim}(M)$ for every finitely generated R-module M.

Proof That $\text{wd}(M) \leq \text{pd}(M)$ follows since projective modules are flat. Take $n < \text{pd}(M)$, then $\text{Ext}^n_R(M,N) \neq 0$ for some $M, N \in R$-mod. The latter group may be embedded in some divisible group G ($\mathbb{Z}$-mod has enough injectives). For this choice of M, N and G we have $\text{Hom}_{\mathbb{Z}}(\text{Ext}^n_R(M,N), G) \neq 0$ and by the previous lemma then the group $\text{Tor}^R_n(M, \text{Hom}_{\mathbb{Z}}(N,G)) \neq 0$, thus $n \leq \text{wd}(M)$. $\square$

7.4.5 Corollary

For any left Noetherian ring R, we have $\text{l.gldim}(R) = \text{wgldim}(R)$. If R is Noetherian then we have : $\text{lgldim}(R) = \text{rgldim}(R) = \text{wgldim}(R)$.

Proof We have $\text{gldim}(R) = \sup\{\text{pd}(C), C$ a cyclic R-module$\} = \sup\{\text{wd}(C), C$ a cyclic R-module$\} \leq \sup\{\text{wd}(M), M \in R\text{-mod}\} = \text{wgldim}(R)$. The other inequality is trivial. Then the left-right symmetry of wgldim finishes the proof.

$\square$

7.4.6 Theorem

Let R be filtered with discrete filtration FR then : $\text{wgldim}R \leq \text{wgldim}(G(R))$.

Proof For an R-module M we may equip it with a filtration such that we have a filt-free resolution : $\ldots \to F_1 \to F_0 \to M \to 0$ such that each $G(F_i)$ is a gr-free $G(R)$-module and the induced complex $\ldots \to G(F_1) \to G(F_0) \to G(M) \to 0$ is a resolution in $G(R)$-gr (see also 7.3.9.). Let N be a filtered right R-module. The sequence $\{N \otimes_R F_i, i \in J\}$ consists of filtered abelian groups with the tensor product filtration. Moreover as each F_i and $G(F_i)$ are free left $R-$, resp. $G(R)$- modules the graded abelian groups $G(N \otimes_R F_i)$ and $G(N) \otimes_{G(R)} G(F_i)$ are isomorphic. This yields that $\text{Tor}^{G(R)}_{.}(G(N), G(M))$ is the homology of the complex :

$$\ldots \to G(N \otimes_R F_1) \to G(N \otimes_R F_0) \to G(M) \to 0$$

If we choose n such that $\text{Tor}^{G(R)}_n(G(N), G(M))$ is zero, then we obtain an exact sequence of abelian groups :

$$(*) \qquad G(N \otimes F_{n+1}) \to G(N \otimes F_n) \to gG(N \otimes F_{n-1})$$

Since we are dealing with finitely generated modules, the filtrations are left limited, so the exactness of (*) leads to the strict exactness of $N \otimes_R F_{n+1} \to N \otimes_R F_n \to N \otimes_R F_{n-1}$, thus we find that $\text{Tor}^R_n(N, M) = 0$.

Consequently if $n > \text{wd}_{G(R)}(G(M))$ then $\text{Tor}^{G(R)}_n(G(M), G(N)) = 0$ and this in turn leads to the fact that $\text{Tor}_n(N, M) \subset 0$ for all right A-modules N. Hence $\text{wd}_R(M) \leq \text{wd}_{G(R)}(G(M))$ for all left R-modules M, or $\text{wgldim}R \leq \text{wgldim}G(R)$. $\square$

The following technical lemma is necessary for our calculation of the gldim of the Weyl algebras.

7.4.7 Lemma

If $A \subset B$ is a ring inclusion such that B is flat as a right A-module then for any left A-module M we have : $\mathrm{wd}_A(B \otimes_A M) \leq \mathrm{wd}_B(B \otimes_A M)$.

Proof If $\ldots F_1 \to F_0 \to M \to 0$ is a flat resolution of M, then by right flatness of B, $\ldots B \otimes_A F_1 \to B \otimes_A F_0 \to B \otimes_A M \to 0$ is a flat resolution of $B \otimes_A M$. For any right A-module N we have $(N \otimes_A B) \otimes_B B \otimes F_i = N \otimes_A (B \otimes_A F_i)$, hence we arrive at :

$$\mathrm{Tor}_*^A(N, B \otimes_A M) = \mathrm{Tor}_*^B(N \otimes_{\mathbb{A}} B, B \otimes_A M)$$

Now, if $k > \mathrm{wd}_B(B \otimes_A M)$ then the right hand side is zero, thus for any right A-module N, $\mathrm{Tor}_k^B(N, B \otimes_A M) = 0$, yielding $\mathrm{wd}_A(B \otimes_A M) \leq \mathrm{wd}_B(B \otimes_A M)$.

□

7.4.8 Application

For the Weyl algebras $\mathbb{A}_n(K)$ we have $\mathrm{gldim}\mathbb{A}_n(K) = n$ $(\mathrm{ch}(K) = 0)$.

Proof

We already know $\mathrm{gldim}\mathbb{A}_n(K) \leq 2n$, hence it is finite (since $\mathrm{gldim}\mathbb{A}(K) \leq \mathrm{gldim}G(\mathbb{A}_n(K)) = \mathrm{gldim}K[X_1, \ldots, Y, \ldots Y_n]$. Now the proof goes by induction on n. If $n = 0$, $\mathbb{A}_n(K) = K$ and the claim is trivial. For $\mathbb{A}_n(K)$ look at the localizations

$$A = K(x_n) \otimes_{K[x_n]} \mathbb{A}_n(K), B = K(y_n) \otimes_{K[y_n]} \mathbb{A}_n(K)$$

Claim $\mathrm{wgldim}A = \mathrm{wgldim}B \leq n$. Indeed, every element of A can be written in the form $\sum f_i y_1^{i_1} \ldots y_n^{i_n}$ with $f_i \in K(x_n)[x_1, \ldots, x_n]$. Define abelian groups $F_n A = \{\sum_{i \leq m} f_i y_n^i, f \in K(x_n) < x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} >$. This defines a filtration on A with associated graded ring $\mathbb{A}_{n-1}(K(x_n))[t]$. By Hilbert's syzygy-theorem and the induction hypothesis $\mathrm{wgldim}(G(A)) \leq n$ and then by the foregoing $\mathrm{wgldim}(A) \leq n$. In a similar way we obtain $\mathrm{wgldim}(B) \leq n$. Now A and B are flat over $\mathbb{A}_n(K)$ since $K(x_n)$, resp. $K(y_n)$, is flat over $K[x_n]$, resp. $K[y_n]$ hence the foregoing lemma applies and we arrive at : $\mathrm{wd}_{\mathbb{A}_n(K)}(A \otimes_{\mathbb{A}_n(K)} M) \leq n$.

Note that any left $\mathbb{A}_n(K)$-module may be embedded into the direct sum of $A \otimes_{\mathbb{A}_n(K)} M$ and $B \otimes_{\mathbb{A}_n K} M$, $M \to (A \otimes M) \oplus (B \otimes M)$ given by $m \mapsto 1 \otimes m + 1 \otimes m$. The latter is zero exactly of $1 \otimes m = 0$ in $A \otimes_{\mathbb{A}_n(K)} M$ and in $B \otimes_{\mathbb{A}_n(K)} M$, hence $f(x_n)m = 0$ as well as $g(y_n)m = 0$ for polynomial f, resp. g. The Weyl algebra $K < x_n, y_n >$ acts on $\mathbb{A}_1(K)m$ but since $fm = 0$, $gm = 0$, the latter is a finite dimensional K-vectorspace. But the Weyl algebra has no finite dimensional modules, hence $m = 0$ and $M \hookrightarrow (A \otimes M) \oplus (B \otimes M)$ is injective. Now assume that $\mathrm{wgldim}\mathbb{A}_n(K) = m > n$. There exist then a left $\mathbb{A}_n(K)$-module M having a minimal flat resolution of length $m > n$. Thus we obtain an exact sequence :

$$0 \to M \to (A \otimes M) \oplus (N \otimes M) \to \overline{M} \to 0$$

The middle term satisfies $\mathrm{wd}((A \otimes M) \oplus (B \otimes M)) \leq n$. For any right $\mathbb{A}_N(K)$-module N we obtain an exact sequence : ($\otimes$ is over $\mathbb{A}_n(K)$) :

$$\mathrm{Tor}_{m+1}(N_1((A \otimes M) \oplus (B \otimes M))) \to \mathrm{Tor}_{m+1}(N, \overline{M}) \to \mathrm{Tor}_m(N, M) \to \\ \to \mathrm{Tor}_m(N.M)$$

Here we have : $\mathrm{Tor}_{m+1}(N, ((A \otimes M) \oplus (B \otimes M))) = \mathrm{Tor}_m(N, ((A \otimes M) \oplus (B \otimes M))) = 0$ thus by exactness : $\mathrm{Tor}_m(N.M) = \mathrm{Tor}_{m+1}(N), \overline{M})$. Now, by choosing an appropriate N we have $\mathrm{Tor}_m(N, M) \neq 0$ and therefore : $\mathrm{wd}_{\mathbb{A}_n(K)}(\overline{M}) \geq m+1 > m = \mathrm{gldim}\mathbb{A}_n(K)$.

The latter is a contradiction. Since wgldim = gldim the claim follows. $\square$

For further theory, e.g. concerning holonomic modules or the use of characteristic varieties in the theory of rings of differential operators, we refer to the literature (cf. [2].

Chapter 8

Solutions to Some Excercises

1.3.1. The Algebra $M_n(K)$

a. The matrices E_{ij} having a 1 only on the place ij and zero elsewhere are generators because they form a K-basis. Consider the permutation matrix $P = E_{n_1} + \sum_{i<n} E_{i,i+1}$. Every $n \times n$-matrix A can now be written as : $P^i E_{11} P^j$, hence two generators suffice. We do need at least two generators since $M_n(K)$, $n > 1$, is not commutative. The relation we need are first $P^n - I$ and $E_{11}^2 - E_{11}$. The relations of the form $E_{ij}E_{kl} = 0$ for $j \neq k$ translate to $E_{11}P^m E_{11} = 0$ for $m < n$ (powers of P on the left and right may be deleted since P is an invertible matrix).

   Finally we rewrite $P = E_{n_1} + \sum_{i<n} E_{i,i+1}$ into $P = \sum_{i=1}^n P^i E_{11} P^{n+1-i}$. These relations suffice because every word in P and E_{11} may be reduced to a linear combination over K of the $P^1 E_{11} P^j$ and these form a basis for $M_n(K)$. Thus we arrive at :

$$M_n(\mathbb{C}) \cong \mathbb{C} < A, P > /(A^2 - A, P^n - I, AP^j A$$

$$\text{for } j < n, P - \sum_{i=1}^{n} P^i A P^{n+1-i})$$

   Remark In stead of writing P as a sum of the $P^i A P^j$ (where $A = E_{11}$) it is also possible to write $I = \sum_{i=1}^n P^i A P^{n-i}$, this relation is equivalent with the other.

b. Left multiplication by a matrix changes rows into a linear combination of the rows. Consequently, the linear space generated by the rows of all matrices in a left ideal I is a subspace of K^n. Conversely, with every subspace V of K^n, we may associate the set of matrices I_V, the rows of which are elements of V. The correspondence $V \to I_V$ is a bijection. By changing from rows to columns the foregoing may be phrased for right ideals. If I is a twosided ideal of $M_n(K)$ with $M \neq 0$ in I, say $M_{kl} \neq 0$ then by suitable right and left multiplications we may transform M into E_{ij} for arbitrary i, j. Hence I contains a K-basis for $M_n(K)$ and thus $I = M_n(K)$.

c. We can embed $\mathbb{C}$ in $M_n(\mathbb{R})$ by mapping $a + bi$, with a. and b. in $\mathbb{R}$ to $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. $M_n\mathbb{R}$ is not a $\mathbb{C}$-algebra because the centre of $M_n(\mathbb{R})$ is $\mathbb{R}$.

1.8.2. The algebra KG

a. A finite abelian group G is $\mathbb{Z}_{p_1} e_1 \oplus \ldots \oplus \mathbb{Z}_{p_k} e_k$ where the p_i are, not necessarily different, prime numbers. The generators of the group are generators of the K-algebra KG, thus $KG = \mathbb{C} < X_1, \ldots, X_k > /(X_i^{e_i} - 1, X_i X_j - X_j X_i)$.

b. The invertible elements of a free algebra or a polynomial ring over K are only $K^* = K - \{0\}$. In any group ring KG there is a K-basis of invertible elements.

c. A rational function continuous in $\mathbb{C}^*$ has a denominator a power of X so we may view this ring as $\mathbb{C}[X, X^{-1}]$ and that is the group algebra for $G = \mathbb{Z}$.

d. Suppose KG is a field and $g \in G$ an element of finite order $g^k = 1$. Then $g - 1$ is a zerodivisor, $(g-1)(1 + g + \ldots + g^{k-1}) = g^k - 1 = 0$. Hence G is an abelian torsion free group. If $u = au_\sigma - bu_\tau$ with $\sigma \neq \tau$ and $a, b \neq 0$ then $u^{-1} = \sum_i a_i u_{g_i}$ for a finite number of nonzero a_i. From $uu^{-1} = 1$ and using that a torsionfree abelian group is ordered, say $\sigma > \tau$, $g_1 > g_2 > \ldots$, then $a_1 u_\sigma u_{g_1} \neq 0$ and $1 = a_1 u_\sigma u_{g_i}$ + lower degree terms yields $1 = a_1 u_\sigma u_{g_i}$ and $d_d u_\tau u_{g_d} = 0$ if g_d is the smallest amongst the g_i. Since we may assume all $a_i \neq 0$, this leads to a contradiction as $u_\tau u_{g_d} \neq 0$.

e. The K-linear map $K(G \times H) \xrightarrow{\psi} KG \otimes_K KH, (g, h) \mapsto g \otimes h$ is a bijection because the $g \otimes h$ for $g \in G, h \in H$, form a K-basis of $KG \otimes KH$. The K-linear isomorphism ψ is also obviously a ring homomorphism.

1.3.3. The exterior algebra

a. Observe that in the exterior algebra we may order all words in the x_i (images of the X_i in $K < X_1, \ldots, X_n >$) by using the anti-commutation relation between them. If $\mathrm{char}(K) \neq 2$ then $2x_i^2 = 0$ so every symbol appears maximally once in a word, hence all words are of the form $x_1^{e_1}, \ldots, x_n^{e_n}$ with $e_i = 0, 1$. The number of possible words is therefore 2^n and this is the K-dimension of the algebra. If $\mathrm{char}(K) = 2$, then the x_i commute and there is no restriction coming from $2x_i^2 = 0$; therefore if $\mathrm{char}(K) = 2$ then the exterior algebra equals the polynomial algebra.

b. All monomials in the x_i are nilpotent because the square of it is zero. Observe that two monomials commute or anticommute depending on the number of x_i that has to be reordered in the product. Look at $S = c_1, M_1 + \ldots + c_k M_k$ where the M_i are monimials different from 1. Then $S^{n+1} = 0$ since every term in S^{n+1} contains at least $n+1$ letters. so some x_i has to appear repeatedly. All elements without constant term are nilpotent and hence zero divisors. An element with constant term nonzero may be written as $\lambda(1 + S)$ for some $\lambda \in K$ and S as before. Now :

$$(1 + S)(1 - S + S^2, \ldots, \pm S^n) = 1 + S^{n+1} = 1$$

implies that $\lambda(1 + S)$ is invertible in the algebra.

c. Since every element with nonzero constant term is invertible a proper ideal I only contains elements of the form $S = aX + bY + cXY$ with $a, b, c \in K$. Suppose a or b is nonzero, then we multiply S by $a^{-1}X$ or $b^{-1}Y$ to obtain dXY hence every ideal contains XY. There are threee possibilities :

1. $I = (XY)$ is 1-dimensional
2. $I = (X, Y, XY)$ is 3-dimensional
3. $I = (aX + bY, XY)$ with a or b nonzero, is 2-dimensional

In case 3. two ideals are equal if the linear terms are a multiple of each other. Therse 2-dimensional ideals are therefore parametrized by the points on a projective line over K.

1.4.3. The Path Algebra

a. A non-trivial path is a composition of arrows $a_1, \ldots, a_k$ such that $s(a_i) = t(a_{i+1})$. A trivial path is a vertex. A $\mathbb{C}$-basis for $\mathbb{C}Q$ is the set of all paths !

b. $\mathbb{C}Q$ is commutative if and only if all arrows are loops.

c. If there are no cycles (these are nontrivial paths which begin where they end).

d. The direct sum of two path algebras is the path algebra of the quiver obtained as the disjoint union of the two quivers. A path in the new quiver is either a path on the first of or a path in the second and the product of paths from the different subquivers is zero.

e. Look at the quiver Q :①→②→ ... → ⓝ. In the path algebra of this there is between every vertex $i \leq j$ a unique path p_{ij}. Define a K-linear map $KQ \to M_n(K), p_{ij} \to E_{ij}$, where E_{ij} is the elementary upper-triangular matrix with 1 only on the place ij, $i < j$. This map is bijective because the p_{ij} are a basis for KQ and the E_{ij} are a K-basis for the uppertriangular matrices. It is easily checked that $KQ \to T_n(K)$ is an isomorphism of K-algebras.

1.3.5. The Quaternion Algebra

a. The relations between i and j allow to rearrange every word in i and j into a word of at most 2 letters. Since $i^2 = j^2 = -1$ and $ij = -ji$ it follows that $1, i, j, ij$ are a set of $\mathbb{R}$-generators. This is also an $\mathbb{R}$-basis because $0 = a_0 + a_1 i + a_2 j + a_3 ij$ leads to $a_i = 0$ $i = 0, \ldots, 3$ because if not then $b = (a_0 - a_1 i - a_2 j - a_3 ij)/(a_0^2 a_1^2 + a_2^2 + a_3^2)$ is an inverse for $a_0 + a_1 i + a_2 j + a_3 ij$.

b. As in a. before every $0 \neq a_i + a_j i + a_2 j + a_3 ij$ is invertible.

c. Choose matrices $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Observe that $AB = -BA$ and $A^2 = B^2 = -I$. Define $\phi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to M_2(\mathbb{C})$ by $\phi(1 \otimes c) = c, \phi(i \otimes c) = cA, \phi(j \otimes c) = cB, \phi(ij \otimes c) = cAB$. One easily verifies that this is a K-algebra isomorphisms from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ to $M_2(\mathbb{C})$.

d. There are are no finite dimensional skewfields over $\mathbb{C}$ as every finite dimensional $\mathbb{C}$-algebra has zero-divisors (see right or wrong questions). Now look at a finite dimensional skewfield Δ over $\mathbb{R}$. Every $x \in \Delta$ then satisfies an irreducible quadratic minimal polynimial over $\mathbb{R}$ because these are the only irreducible polynomials over $\mathbb{R}$. Look at the set S of elements in Δ having minimal polynomial of type $X^2 + \lambda$ with $\lambda \in \mathbb{R}^+$. If $x \in S$ and $r \in \mathbb{R} - \{0\}$ then $rx \in S$ but $r + x \notin S$ because the minimal polynomial of $r + x$ is $X^2 - 2rX + \mu$. If $u, v \in S$ then also $(u + v)$ and $(u - v) \in S$. Assume that $u + v \notin S$ then u, v and 1 are linearly independent because otherwise $u = av + b$ with $a, b \in \mathbb{R}$, $a \neq 0$ and we may assume $b \neq 0$ because otherwise $u = av$ and the claim holds; then $av \in S$ yields $av + b \notin S$ contradicting $u \in S$. Since $u + v, u - v$ satisfy a quadratic polynomial we obtain : $(u+v)^2 = p(u+v)+q$, $(u-v)^2 = r(u-v)+s$. Put $u^2 = c, v^2 = d$ with $c, d \in \mathbb{R}^-$. Then :

$$uv + vu = p(u + v) + q - c - d$$
$$uv + vu = -r(u - v) - s + c + d$$

Thus $(p+r)u + (p-r)v + (q+s-2c-2d) = 0$. Since $u, v, 1$ are $\mathbb{R}$-linearly independent we must have : $p + r = p - r = 0$, hence $p = r = 0$, hence $(u + v)^2 - q = 0$ and thus $u + v \in S$. Similarly $u - v \in S$. Therefore S is an $\mathbb{R}$-vectorspace and every element of A can uniquely be written as $r+x$ with $r \in \mathbb{R}, x \in S$. We may define on S a positive definite metric :$g(u, v) = -(u + v)^2 + u^2 + v^2 = -(uv + vu)$. Let $\{b_1, \dots, b_n\}$ be an orthogonal basis for S (observe that elements of this basis anticommute). Since A is noncommutatitve $\dim_{\mathbb{R}} S > 1$. We have that $\dim_{\mathbb{R}} S \geq 3$ because $g(b_1, b_1b_2) = g(b_2, b_1b_2) = 0$ and $(b_1b_2)^2 = -1 \neq 0$. If $\dim S = 3$ then $A \cong \mathbb{H}$.

Assume that $\dim_{\mathbb{R}} S > 3$; then there is an orthogonal basis $\{b_1, b_2, b_1b_2, b_4, \dots\}$ but since b_4 anticommutes with b_1 and b_2 we must have $(b_1b_2)b_4 = -(b_1b_4b_2) = b_4(b_1b_2)$ but that contradicts $g(b_1b_2, b_4) = 0$.

1.3.6. Clifford Algebras

a. The definition will be independent of the choice of basis if for every pair of vectors $a = \sum a_ie_i$ and $b = \sum b_jc_j$ we have that :

$$\begin{aligned} ab + ba &= (\sum a_ie_i)(\sum b_je_j) - (\sum b_je_j)(\sum a_ie_i) \\ &= \sum_{i,j} 2a_ib_jg(e_i, e_j) \\ &= 2g(\sum a_ie_i, \sum b_je_j) = 2g(a, b) \end{aligned}$$

There is up to isomorpism only one Clifford algebra of dimension n since every metric is isomorphic to the standard metric on $\mathbb{C}$.

b. Each word in the generators can contain a basis element e_i at most one time because $e_i^2 = -g(e_i, e_i)$. We can order the words "alphabetically" modulo lower degree terms.

Argumentation like the one used for the exterior algebra now yields that the dimension is 2^n.

c. We have $C(\mathbb{C}^2, g) = \mathbb{C} < X, Y > /(X^2 + 1, Y^2 + 1, XY + YX) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{C})$.

d. We have

$$\begin{aligned} &C(\mathbb{R}, 1) = \mathbb{R}[X]/(2X^2 + 2) = \mathbb{C}, C(\mathbb{R}, -1) = \mathbb{R}[X]/(2X^2 - 2) = \mathbb{R} \oplus \mathbb{R} \\ &C(\mathbb{R}^2, 1) = \mathbb{R} < X, Y > (X^2 + 1, Y^2 + 1, XY + YX) = \mathbb{H} \\ &C(\mathbb{R}^3, 1) = R < X, Y, Z > (X^2 + 1, Y^2 + 1, Z^2 + 1, \\ &\qquad\qquad\qquad\qquad XY + YX, XZ + ZX, YZ + ZY) \end{aligned}$$

Put $W = \frac{\sqrt{2}}{2}(1 + xyz)$, x, y and z being the images of X, Y, Z. Then W is a central idempotent hence $C(\mathbb{R}^3_, 1) = C(\mathbb{R}^3, 1)W \oplus C(\mathbb{R}^3, 1)(1 - W)$ and $WC(\mathbb{R}^3, 1) = \mathbb{R} < \xi, \eta > /(\xi^2 - \eta^2, \xi\eta + \eta\xi) = \mathbb{H}$, where $\xi = xW$, $\eta = yW$, W is the unit in $WC(\mathbb{R}^3, 1)$ and $zW = (\sqrt{2} - 1)xWyW$. Similarly $(1 - W)C(\mathbb{R}^3, 1) = \mathbb{H}$, hence $C(\mathbb{R}^3, 1) \cong \mathbb{H} \oplus \mathbb{H}$.

1.4. Right or Wrong ?

1. *Right.* Such an algebra is an epimorphic image of $K[X]$.

2. *Wrong.* For example $R = \mathbb{C}[X, Y]/(X^2, Y^2, XY)$ is not generated by one element. For each element $c = a_0 + a_1x + a_2y$ the ring $\mathbb{C}[c]$ is two-dimensional because : $c^2 = a_0^2 + 2(a_1x + a_2y) = 2c + a_0^2 - a_0$. But R itself has dimension 3 over $\mathbb{C}$.

3. *Wrong.* $\mathbb{C}[X]$ is generated by X but $\mathbb{C}[X^2, X^3]$ cannot be generated by one element.

4. *Right.*

   If A is generated by $a_1, \ldots, a_n$ then $\phi(A)$ is generated by $\phi(a_1), \ldots, \phi(a_n)$.

5. *Right.* We prove this first for $\mathbb{C} < X_1, \ldots, X_n > /(X_iX_j, 1 \leq i, j \leq n)$. Let R be the latter ring, for every two elements a and b in R we have $ab = \lambda, a + \lambda_2 b + \lambda_3$ with $\lambda_i \in \mathbb{C}$. Suppose R is generated by $a_1, \ldots, a_m$, m minimal as such; then $1, a_1, \ldots, a_m$ is a $\mathbb{C}$-basis for R. Thus a set of generators for R contains at least n elements. Now R is the image of the free algebra $\mathbb{C} < X_1, \ldots, X_n >$ hence every set of generators for $\mathbb{C} < X_1, \ldots, X_n >$ contains at least n elements, using the foregoing question 4.

6. *Right.* If $a = a_0 + a_1w_1 + \ldots + a_nw_n \in K < \mathcal{X} >$ is invertible with inverse $b = b_0 + b_1w'_1 + \ldots + b_mw'_m$ where the w_i, w'_j are words ordered by ascending length and $a_i, b_j \neq 0$. Then $1 = ab = a_0b_0 + \ldots + a_nb_mw_nw'_m$, thus $a_nb_m = 0$ but that is a contradiction.

7. *Wrong.* Suppose that $r_i = f_i(x)g_i(x), 1 \leq i \leq n$ are algebra generators. Then $(1 + g_1(X)g_2(X) \ldots g_n(X))^{-1}$ has to be written as a polynomial expression in the a_i, hence it would be writable as a function of polynomnials with common denominator $g_1(x) \ldots g_n(x)$, this is impossible (as is easily seen).

8. *Wrong.* There is no map of $\mathbb{R}$-algebras $\mathbb{C} \to \mathbb{R} < X_1, \ldots, X_n >$ because it would have to be injective but the only invertible elements of $\mathbb{R} < X_1, \ldots, X_n >$ are those of $\mathbb{R} - \{0\}$.

9. *Wrong.* $\mathbb{C}[X]/(X^2) \to \mathbb{C}, X \mapsto 0$ is a surjective morphism.

10. *Right.* Let $a \in R$ be a zero divisor and consider aR. This is an ideal as R is commutative and $1 \notin aR$, thus a R is a nontrivial ideal.

11. *Right.* We have $K < \mathcal{X} > *K < Y >= K < \mathcal{X} \cup Y >$.

12. *Wrong.* $K[X] * K[Y] = K < X, Y >$.

13. *Right.* Because $(a, b)(c, d) = (ac, bd) = (ca, db) = (c, d)(a, b)$.

14. *Right.* Such algebras are generated by one element.

15. *Wrong.* $\mathbb{C}[X]/(X^2) * \mathbb{C}[Y]/(Y^2) = \mathbb{C} < X, Y > /(X^2, Y^2)$ and the latter has $\mathbb{C}$-basis $1, X, Y, XY, YX, XYX, YXY, \ldots$.

16. *Wrong.* $\mathbb{C}[X]/(X^2) \oplus \mathbb{C}$ has $(X, 0)$ as nilpotent.

17. *Wrong.* $\mathbb{H} \otimes \mathbb{C} = M_2(C)$, see examples.

18. *Right if complex.*

    An arbitrary 2-dimensional complex algebra is $\mathbb{C}[X]/(X^2 + aX + b)$. By passing to $Y = CX + d$ this is either $\mathbb{C}[Y]/(Y^2)$ or $\mathbb{C}[Y]/(Y^2 - Y)$ according to whether $X^2 + aX + b$ has a double zero or not. In the first case the algebra has nilpotents; in the second case it is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. If the algebra is not complex then it is wrong ! Then $\mathbb{C}$ is a two dimensional algebra over $\mathbb{R}$ without zerodivisors or nilpotents.

    Right. Because then there is a surjective isomorphism of $K < X >$ to that algebra and $K < X >= K[X]$ is commutative.

19. *Right.* If e is idempotent in R, then also $1 - e$ and eR may be viewed as a ring with unit e (similar for $(1 - e)R$). Look at $\pi : eR + (1 - e)R \to R : (a, b) \mapsto a + b$. This is a morphism since $e(1 - e) = 0$, hence $(a + b)(c + d) = ac + ad + bc + bd = ac + bd$, where $a, c \in eR$ and $b, d \in (1 - e)R$. Also π is surjective as $a = ea + (1 - e)a$ and π is injective because $ea + (1 - e)b = 0 \Rightarrow e(ea + (1 - e)b) = e^2a = ea = 0$ and $(1 - e)b = 0$.

20. *Wrong.* Take $\phi_A = \phi_B : \mathbb{C} \to \mathbb{C}$ the identity, then $\phi_A \oplus \phi_B$: $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ cannot be surjective as $\mathbb{C} \oplus \mathbb{C}$ is 2-dimensional.

21. *Wrong.* Put $A = M_n(\mathbb{C})$; that algebra has no nontrivial ideals but it has nilpotents (a.o. the upper triangular matrices). Nilpotents do not form a left (nor right) module, for example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is not nilpotent.

22. *Right.* Look at $M_n(R)$ where n is the number of vertices and map the i^{th} vertex on the i^{th} diagonal component, the arrow from vertex i to vertex j is mapped to the matrix E_{ij} with 1 on the entry i, j and zero elsewhere. One easily checks that between the arrows and the corresponding matrix the same relations hold (because there are no relations $E_{ij}E_{ji} = E_{ii}$ as there it at most one arrow between i and j).

23. *Wrong.* Let Q be a quiver such that $\mathbb{C}Q = M_n(\mathbb{C})$. Look at the B-vector space spanned by the paths of length at least one. This is an ideal of $\mathbb{C}Q$ but the matrix algebra does have nontrivial ideals.

24. Put $R = \mathbb{C} < X_1, \ldots, X_n >$, $m = (X_i, 1 \leq i \leq n)$ and look at the canonical $\pi_n : R \to R/m^n$ and the $\mathbb{C}$-linear map $\iota_n : R/m^n \to R$, $w \bmod m^n \mapsto w$, where the w are words of length smaller than n (the ι_n are not ring morphisms !) If $\phi : \mathbb{C} < X_1, \ldots, X_n > \to \mathbb{C} < X_1, \ldots, X_n >$ is surjective then $\pi_n \phi \iota_n$ is also surjective. Since $\pi_n \phi \iota_n$ is a $\mathbb{C}$-linear map between finite dimensional $\mathbb{C}$-spaces of the same dimension, $\pi_n \phi \iota_n$ also must be injective. Suppose $x \neq 0$ is in $\mathrm{Ker}\phi$ and the longest word appearing in x has length smaller than n then $x \bmod m^n$ is in $\mathrm{Ker}\pi_n \phi \iota_n$ and therefore $x \bmod m^n \neq 0$ contradicts injectivity of the latter K-linear map.

25. *Wrong.* $\mathbb{C}[X, Y].(XY - 1)$ is not a field e.g. $(1 - X)$ has no inverse.

26. *Right.* Put $b_i = \sum_j M_{ij} a_j$ then $a_j = \sum_i (M^{-1})_{ji} b_i$; the bj's generate the a_j's.

27. *Wrong.* Let ϕ be an automorphism, then $\phi(X)$ must have degree one, hence $\phi(X) = X$ or $\phi(X) = X + 1$.

28. *Right.* If $|A| < \infty$ then A is a finite dimensional vector space over a finite field, so it has $|F^m| = (p^n)^m$ elements.

29. *Right.* If A is such an algebra and $x \in A - K$ then $K[x]$ is a subalgebra, thus $A = K[x]$ is commutative.

30. *Wrong.* $K[X]/(X)^2$ has no nontrivial subfields as it is a two-dimensional K-vector space.

31. *Right.* Let a and b be those nilpotents, say $a^n = b^m = 0$. Since every element is a linear combination of monomials $a^i b^j$ and only a finite number of these are nonzero ($i < n, j < m$ must hold) the dimension of the algebra is smaller than or equal to mn.

32. *Wrong.* $K[X]$ has a subalgebra $K[X^n]$ for every $n \in \mathbb{N}$.

33. *Wrong.* In $\mathbb{F}_2 < X, Y >$ we have :

$$(X + Y)^2 = X^2 + Y^2 + XY + YX \neq X^2 + Y^2$$

34. *Right.* The bijection is $(\phi : K < \mathcal{X} > \to A) \mapsto \phi|\mathcal{X}$. This is an injection since a K-morphism is uniquely determined by the images of the generators, it is also surjective since $K < \mathcal{X} >$ is the free algebra, therefore every set of images of a set of free generators, e.g. $\mathcal{X}$, comes from a fitting K-algebra morphism.

35. *Right.* Put $\phi : \mathbb{A}_1(\mathbb{C}) \to M_n(\mathbb{C})$ a $\mathbb{C}$-algebra morphism then $\phi(x)\phi(y) - \phi(y)\phi(x) = I$. Taking trace at both sides yields
$$\mathrm{Tr}(\phi(x)\phi(y)) - \mathrm{Tr}((\phi(y))(\phi(x))) = \mathrm{Tr}I = n$$
but the left hand side is zero, contradiction.

36. *Right.* Put $a \in A - \mathbb{C}$ then a is algebraic. Say $f(a) = 0$. Split $f(X) = (X - \lambda_1) \dots (X - \lambda_r), \lambda_i \in \mathbb{C}$, then $f(a) = 0$ and $a \neq \lambda_i, i = 1, \dots, r$ yields that $a - \lambda_i$ is a zero divisor.

37. *Right.* Let $\phi : A \to A$ be a morphism. From $\dim A - \dim \mathrm{Ker}\phi = \dim \mathrm{Im}\phi$ it follows that : $\mathrm{Ker}\phi = 0$ entails $\dim A = \dim \mathrm{Im}\phi$, or ϕ is bijective.

38. *Wrong.* The sum of algebra endomorphisms is not an algebra endomorphism : $(\phi + \psi)(1) = \phi(1) + \psi(1) = 2 \neq 1$!

39. *Right.* Let A be n-dimensional, then for every $a \in A$ we have a linear map $\phi_a : A \to A, x \mapsto ax$. Choose a basis for A and let M_a be the $n \times m$-matrix corresponding to ϕ_a. Then it is easy to check that $M_{n+b} = M_a + M_b$ and $M_{ab} = M_a M_b$. The map $a \mapsto M_a$ defines a morphism $A \to M_n(R)$. This morphism is injective because $M_a = 0$ entails $ax = 0$ for all x, in particular for $x = 1$, therefore $a = 0$ follows.

40. *Right.* Define $\phi : M_n(K) \otimes M_m(K) \to M_{mn}(K)$, $E_{ij} \otimes E'_{kl} \mapsto E''_{im+k,jm+l}$, where $E_{ij}, E'_{k,l}$ and $E''_{\nu,\mu}$ are elementary matrices with exactly one 1 in the specified entry and zero elsewhere.

41. *Right.* The quaternion algebra does not have proper ideals as every nonzero element is invertible.

42. *Wrong.* $\mathbb{C} < X_1X_2, X_2X_3 > \subset \Lambda_3(\mathbb{C})$ is commutative but not generated by 1 element as it is isomorphisc to $\mathbb{C}[X,Y]/(X^2, Y^2, XY)$.

43. *Wrong.* $\mathcal{C}(C^2, 1) = \mathbb{C} < X, Y > /(XY + YX, X^2 - 1, Y^2 - 1)$ hence $(X(1+Y))^2 = X(1+Y)X(1+Y) = X^2(1+Y)(1-Y) = X^2(1-Y^2) = 0$.

44. *Right.* The map $\Lambda_n \to \Lambda_{n-1}, X_i \to X_i, X_n \to 0$ is surjective.

45. *Right.* The map $\Lambda_{n-1} \to \Lambda_n, X_i \to X_i$ is an embedding.

46. *Wrong.* $G(\mathbb{C}^2, 1)$ does not have natural ideals because $C(\mathbb{C}^2, 1) \cong M_2(\mathbb{C})$, thus it cannot be mapped onto $\mathcal{C}(C, 1) = \mathbb{C}[X]/(X^2)$.

47. *Right.* A path-algebra without zero divisors has only one vertex and such a path-algebra has no relations between the arrows.

48. *Wrong.* The path algebra of the quiver with two vertices and in each vertex a loop has zero divisors but no nilpotents.

49. *Right.* The images of the vertices are either zero or one. As the sum of all vertices is one there can only be one that is mapped to one, the others must be mapped to zero. An arrow arriving or starting in a vertex mapped to zero is mapped itself to zero, hence only loops may be mapped to one.

50. *Wrong.* The idempotents of a path-algebra can be written as sums of vertices. Therefore all idempotents commute. Let Q be the quiver with 2 vertices and no arrows. Then $\mathbb{C}\mathbb{Q} \otimes \mathbb{C}Q$ cannot be a path algebra because idempotents from the first component do not commute with idempotents from the second component.

2.2.1.

In $M_2(\mathbb{C})$ we calculate : $[E_{11}, E_{22}] = 0$, $[E_{11}, E_{12}] = E_{12} = -[E_{22}, E_{12}]$, $[E_{22}, E_{21}] = E_{12} = -[E_{11}, E_{21}]$ and $[E_{12}, E_{21}] = E_{11} - E_{22}$. Lie subalgebras of dimension one are CM with M some matrix. Which of these are in fact ideals ? Put $M = m_{ij}E_{ij}$, then : $[E_{11}, M] = m_{12}E_{12} - m_{21}E_{21} = cM \in \mathbb{C}M$. If M is not diagonal (otherwise we may take $c = 0$) then M is either upper-triangular (only $m_{12} \neq 0$) or lower triangular (only $m_{21} \neq 0$) depending on whether $c = +1$ or -1. Both cases excluded since $[E_{12}, E_{21}] = E_{11} = E_{22}$. If M is diagonal then it is a scalar because we have $[E_{12}, M] = (m_{22} - m_{11})E_{12}$, i.e. $m_{22} = m_{11}$ follows. There is only one one-dimensional Lie - ideal i.e. $\mathbb{C}.I$. What are the two-dimensional Lie-subalgebras. Either it is Abelian or else there exist 2×2-matrices such that $[M, N] = M$. In the first case we are looking for two commuting K-linearly independent M, N such that $[M, N] = 0$. By conjugating M by an invertible matrix we can bring M to its Jordan normal form M' which is diagonal or of the form $\lambda I + E_{12}$. There are 3-possibilities :

a. $M = M' = \lambda I$; then we may choose any nonscalar N since I and $N + \nu I$ generates the same Lie algebra as I and N we may choose N with $\mathrm{Tr}N = 0$. Such a Lie algebra is an ideal if $[N, E_{ij}] = \lambda_1 + \lambda_2 N$. As $\mathrm{Tr}[N, E_{ij}] = 0$ we must have $\lambda_1 = 0$ and thus KN is also a Lie ideal, contradiction. Thus these Lie subalgebras are not ideals.

b. $M' = \lambda_1 E_{11} + \lambda_2 E_{22}$ with $\lambda_1 \neq \lambda_2$. Then $[M, N] = (\lambda_1 - \lambda_2)n_{12}E_{12} - (\lambda_1 - \lambda_2)n_{21}E_{21} = 0$, thus N must be diagonal. The Lie subalgebra is thus 2-dimensional but so is the Lie algebra of diagonal matrices. The latter contains I and we are in the first case.

c. M and N have double eigenvalues and are not scalar. In this case : $[M', N] = [\lambda I + E_{12}, N] = (n_{11} - n_{22})E_{12} + n_{21}(E_{11} - E_{22})$ and therefore $N = \nu I + \epsilon E_{12}$. Taking the suitable linear combination of M' and N we arrive again in the first case. Hence all Lie subalgebras of dimension 2 are of the form $\mathbb{C} + \mathbb{C}N$ where N is an arbitrary nonzero matrix with $\mathrm{Tr}N = 0$. None of these Lie subalgebras is a Lie ideal.

   Next look at at 3-dimensional Lie subalgebras ! Any 3-dimensional Lie subalgebra T may be represented by a linear equation in the coefficients of the matrices. By a suitable choice of a matrix S we may present this as $X \in T \Leftrightarrow \mathrm{Tr}(XS) = 0$ (verify this). With every S there corresponds a subspace and we want to describe when this subspace is a Lie subalgebra. Hence, $\mathrm{Tr}XS = \mathrm{Tr}YS = 0 \Rightarrow \mathrm{Tr}([X, Y])S) = 0$.

Again we may bring S is Jordan form and again we may consider three cases.

a. S is scalar, then T is the space of trace zero matrices in this case T is a Lie subalgebra because commutators have zero trace. Then T is also obviously a Lie ideal.

b. S has two different eigenvalues, say λ_1 and λ_2. Then T is the space of $t(\lambda_2 E_{11} - \lambda_1 E_{22}) + lE_{12} + mE_{21}$; taking the commutator with $E_{12} \in T$ leads to a condition $(l-m)(E_{11} - \mathbb{E}_{22})$ + *"off diagonal"* but this is not a Lie subalgebra.

c. If S is not scalar but has a double eigenvalue, then T consists of : $kE_{11} + lE_{22} - \lambda(k+l)E_{12} + mE_{21}$. Taking the commutator with $E_{11} - E_{12}$ leads to a condition on the elements : $-\lambda(k+l-k+l)E_{12} - mE_{21}$, which is not in T.

The only 3-dimensional Lie subalgebra, and Lie ideal, is the one of trace zero matrices.

2.2.3.

a. If $f, g : \mathbb{R} \to G$ are differentiable functions then $X_{f(s)gf^{-1}(s)}$ is for every $s \in \mathbb{R}$ a vector in g. The map $s \mapsto X_{f(s)gf^{-1}(s)}$ is a differentiable function in the vector space g, thus also

$$\frac{d}{ds} X_{f(s)gf^{-1}(s)xi}|_{s=0} \in g$$

But we have

$$\frac{d}{ds} X_{f(s)gf(s)^{-1}}|_{s=0} = \frac{d^2}{dsdt} f(s)g(t)f^{-1}(s)$$
$$= X_f X_g f^{-1}(0) - f(0)XgX_f f^{-2}(0) = [X_f, X_g]$$

b. Since for small enough ε and arbitrary M we have : $(1+\varepsilon M)^{-1} = (1 - (\varepsilon M) + (\varepsilon M)^2 - (\varepsilon M)^3 + \ldots)$. Therefore M is a tangent vector to $1 \in \mathrm{GL}_n(\mathbb{R})$, in other words $\mathrm{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$.

For $\det(1+\varepsilon M) = 1 + \varepsilon \mathrm{tr}(M) + O(\varepsilon^2)$, thus if $\det(1+\varepsilon M) = 1$ we must have $\mathrm{tr}(M) = 0$; therefore $\mathrm{sl}_2(\mathbb{R})$ consists of matrices with trace zero.

For $(1+M)^t = (1+\varepsilon M)^{-1} = 1 - (\varepsilon M) + (\varepsilon M)^2 - \ldots$, we must have that $M^t = -M^t$, thus $\mathrm{so}_n(\mathbb{R})$ consists of antisymmetric matrices.

c. Consider the Levi-Civita tensor :

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } ijk \text{ is an even permutation of } 123 \\ -1 \text{ if } ijk \text{ is an odd permutation of } 123 \\ 0 \text{ if } ijk \text{ is not a permutation of } 123 \end{cases}$$

Observe that : $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ilj}$. Let $\overline{e}_{i,i} = 1, 2, 3$, be the unit vectors in $\mathbb{R}^3$ then :

$$\overline{e}_i \times \overline{e}_j = \varepsilon_{ijk} e_k \text{ (summation over } k)$$

The matrices $\varepsilon_{1ij}, \varepsilon_{2ij}$ and ε_{3ij} are antisymmetric 3×3-matrices forming a basis for $so_3(\mathbb{R})$. Now : $\varepsilon_{ijk}\varepsilon_{imk} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$, thus we obtain :

$$\begin{aligned} [\varepsilon_{X\ldots}, \varepsilon_{Y\ldots}] &= \varepsilon_{X_{ij}}\varepsilon_{Y_{jk}} - \varepsilon_{Y_{ij}}\varepsilon_{X_{jk}} \\ &= \delta_{Xk}\delta_{Yi} - \delta_{XY}\delta_{ik} + \delta_{YX}\delta_{ki} - \delta_{Yk}\delta_{Xi} \\ &= \delta_{Xk}\delta_{Yi} - \delta_{Yk}\delta_{Xi} = \varepsilon_{XYj}\varepsilon_{jik} \end{aligned}$$

Up to changing the dummy-indices now we see that the structure constants for $SO_3(\mathbb{R})$ and $\mathbb{R}_3, \times$ are the same they are isomorphic Lie algebras.

d. If N is a normal subgroup of a Lie group and $f : \mathbb{R} \to G$, $g : \mathbb{R} \to N$ are differentiable, then for all $t, s \in \mathbb{R}$ we have $f(s)g(t)f^{-1}(s) \in N$, thus $[X_f, X_g] = \frac{d}{ds} X_{f(s)gf^{-1}(s)}|_{s=0}$ is in N too.

2.2.4.

a. Let g be a complex Lie-algebra of dimension 2 that is not Abelian. Choose $x \in g$ and look at the map $y \mapsto [x, y]$. This has an eigenvector $z \neq \lambda x$, the space $\mathbb{C}z$ is then a Lie ideal so g is not simple.

b. Write E for the matrix with a 1 in the upper right hand corner, F for the matrix with $a+1$ in the lower left hand corner, K for the matrix with 1 and -1 in the diagonal (zero elsewhere in each case).

Assume that $\mathrm{sl}_2(\mathbb{C})$ has a Lie ideal containing some $V = a_1 E + a_2 F + a_3 K$. If a_1, a_2 are both zero then K is in the ideal and then also $[K, E] = 2E, [K, F] = -2F$ and the ideal contains everything. Up to replacing V by $W = [K, V]$ in case a_1, a_2 are not both zero, we may assume $a_3 = 0$. If only $a_1 = 0$, then $[E, W] = \lambda K$ is in the ideal and then again the ideal contains everything (as before); similar for $a_2 = 0$. So we have to deal with $a_1 \neq 0, a_2 \neq 0$, then W and $[K, W]$ are linearly independent. So we may write E and F as linear combinations of W and $[K, W]$; therefore the ideal contains E and F and then also $K = [E, F]$. Consequently $\mathrm{sl}_2(\mathbb{C})$ has no nontrivial Lie ideals.

c. We can embed $\mathrm{sl}_2(\mathbb{C})$ in two ways as a Lie subalgebra of $sl_3(\mathbb{C})$; first in the first two rows and columns, secondly in the last two rows and columns. Using the basis given in b. we may define E_i, F_i, K_i for $i = 1, 2$. We know $\mathrm{sl}_3(\mathbb{C})$ has dimension 8, so how can we express the two remaining basiselements in the $E_i, F_i K_i$? The matrix with a 1 in the upper right corner is $[E_1, E_2]$, the matrix with a 1 in the lower right hand corner is $[F_2, F_4]$.

2.2.6.

Put $E_i = A_{i+1,i}, F_i = A_{i,i+1}$ and $K_i = A_{i,i} - A_{i+1,i+1}$. It is now easy enough that all relations for the enveloping algebra hold for these choices.

2.4.13.

Take

$$P(T) = \sum_{i=1}^{k+1} \frac{c_i(T - a_1) \dots \widehat{(T - a_i)} \dots (T - a_{k+1})}{(a_i - a_1) \dots \widehat{(a_i - a_i} \dots (a_i - a_{k+1})}$$

where the symbol $\wedge$ above a term means that the term under it is deleted.

2.4.14.

$\mathrm{Tr}([x, y]z) = \mathrm{Tr}(xyz - yxz) = \mathrm{Tr}(xyz) - \mathrm{Tr}(y(xz)) = \mathrm{Tr}(xyz) - \mathrm{Tr}((xz)y) = \mathrm{Tr}(x[y, z])$.

2.4.16.

Let (V_i) be a flag stabilized by g, take $x \in g, y \in [g,g]$. Computing $\mathrm{Tr}(xy)$ in the basis given by the flag considered we obtain that x is upper triangular, y is strictly upper triangular, so all elements on the diagonal of xy are zero, hence $\mathrm{Tr}(xy) = 0$.

2.4.20.

Let $\{w_1, \ldots, w_k\}$ be a K-basis for W which we complete to a K-basis of V, say $\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}$. From the hypothesis it follows that : $\varphi(w_i) = \sum_{j=1}^{k} a_{ji} w_j$ and $\varphi(v_j) \in W$. Consequently the matrix of φ in the chosen basis has the form $\begin{pmatrix} (a_{ij}) & (\ \ ) \\ (0) & (0) \end{pmatrix}$, where $(\ldots)$ are blocks of the suitable size. Computing traces of φ and $\varphi|W$ in the above basis yields the assertion.

2.4.23.

Clearly g is semisimple $\Leftrightarrow$ there are no nonzero solvable ideals. Since any abelian ideal is solvable, the implication : g is semisimple $\Rightarrow$ there are no abelian ideals is easy. Conversely, assume that g is not semisimple, then $\mathrm{Rad}(g)$ is nonzero and solvable. The last nonzero term in the derived series for $\mathrm{Rad}(g)$ is then a nonzero abelian ideal of g,

2.4.26.

A 1-dimensional subspace of $\mathrm{sl}_2(K)$ is not semisimple. For the second part establish that any derivation of a semisimple g is inner (see Theorem 2.3.33).

2.4.28

Put $\beta_{ik} = \beta(e_i, e_k)$. We look for a matrix (ν_{kj}) such that the $f_i = \sum_{k=1}^{n} \alpha_{kj} e_k$ form a basis, i.e. $\beta(e_i, \sum_{k=1}^{n} \alpha_{kj} e_k) = S_{ij}$. But $\beta(e_i, \sum_{k=1}^{n} \alpha_{kj} e_k) = \sum_{k=1}^{n} \beta_{ik} \alpha_{kj}$ and so the matrix (α_{kj}) is the inverse of (β_{kj}).

2.4.31

Let $\{E, F, K\}$ be the standard basis of $\mathrm{sl}_2(K)$. The dual basis relative to the trace form is $\{K, \frac{1}{2}F, E\}$. Thus $c_\rho = EK + \frac{1}{2}F^2 + KE = \frac{3}{2}I = (\dim g / \dim V)I$, I being the identity matrix.

2.4.32.

Consider a one-dimensional g-module K_v. For every $x \in g$ there is an $a_x \in K$ such that we have : $x.v = a_x v$. Since g is semisimple, $g = [g,g]$ hence it is enough to establish that $xv = 0$ for all $x \in [g,g]$ i.e. for $x = [y,z]$. However $[y,z]v = yz.v - zy.v = a_y a_x v - a_x a_y v = 0$.

2.4. Right or Wrong

1. *Right.* If a ring is not commutative and a does ot belong to the centre, then $[a, -]$ is a nontrivial derivation.

2. *Wrong.* Take $K[X]/(X^2)$ then $d : a + bX \mapsto bX$ is a K-derivation and $d^2 = d$.

3. *Right.* Because $d(i^2) = 0 = 2idi$ yields $di = 0$ and $d = 0$.

4. *Right.* Let d be a derivation, we look at dA_{ij} with A_{ij} the standard basis. Since $A_{12}^2 = 0$ we must have : $A_{12}(dA_{12}) = (-dA_{12})A_{12}$. This means that the left under entry of dA_{12} is zero and the diagonal elements are opposite. Similar for dA_{21} but then with a zero in the right upper corner. Now $(A_{12}+A_{21})^2 = I$, thus $0 = (A_{12}+A_{21})d((A_{12}+A_{21})^2) = d(A_{12}+A_{21}) + (A_{12}+A_{21})d(A_{12}+A_{21})(A_{12}+A_{21})$. Left multiplication by $A_{12}+A_{21}$ interchanges the rows, right multiplication by $A_{12}+A_{21}$ interchanges the columns. The second term is thus $d(A_{12}+A_{21})$ with the diagonal elements interchanged and the off-diagonal elements interchanged. Consequently, the sum of the off-diagonal elements in $d(A_{12}+A_{21})$ must be zero. Therefore : $dA_{12} = aA_{11} - aA_{22} + cA_{12}, dA_{21} = bA_{11} - bA_{22} - cA_{12}$. Since $A_{12}A_{21} = A_{11}, A_{21}A_{12} = A_{11}$, we know the derivation is uniquely determined by $a, b, c,$. For every $a.b.c$ there is such a derivation, e.g. the commutator with $\begin{pmatrix} 0 & c \\ a & b \end{pmatrix}$

5. *Wrong.* In $K[X]$ all inner derivations are zero but $X - X^2$ is not in K.

6. *Right.* Because $2I = 0$.

7. *Right.* Suppose A is n-dimensional. Since every derivation is linear we may embed $\mathrm{Der}_K A$ into $M_n(K)$, thus $\dim_K \mathrm{Der}_K A \leq n^2$.

8. *Wrong.* The inner derivations of $A = \mathbb{R}[X] \oplus \mathbb{H}$ form a 3-dimensional vector space, $\mathrm{Inn}_{\mathbb{R}}(A) = A/Z(A) = \mathbb{R}[X]/Z(\mathbb{R}[X]/ \oplus \mathbb{H}/Z(\mathbb{H}) = \mathbb{H}/\mathbb{R}$ and this is 3-dimensional.

9. *Wrong.* For $R = K < X, Y >$ the derivation mapping X to Y and Y to 0 is d say. Then $d(X^2) = XY + YX$.

10. *Right.* Let λ be a generator of $\mathbb{F}_q$ such that $\lambda^{q-1} = 1$, then $d\lambda^{q-1} = 0$, hence $(q-1)\lambda^{q-2}d\lambda = 0$. Since $q-1$ is not zero modulo $p, d\lambda = 0$ follows.

11. *Right.* The derivations form a subspace of $M_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the prime field of the algebra.

12. *Wrong.* $\mathrm{Der}_{\mathbb{R}}\mathbb{C} = \mathrm{Der}_{\mathbb{R}}\mathbb{R} = 0$.

13. *Right.* See theory.

14. *Right.* If $da = db = 0$ then $d(a+b) = 0$ and $d(ab) = 0$.

15. *Right.* Define a left A-module structure on $\mathrm{Der}_K A$ by $ad : A \to A, x \mapsto adx$. Observe that this does not work in case A is non-commutatitve.

16. *Wrong.* If $d : \mathbb{C}[X] \to [X]$ is a derivation then we could try to define d' on $\mathbb{C}[X]/(X^2)$ by putting $d'(a + (X^2)) = da + (X^2)$. This is possihble only if $d(X^2) \subset (X^2)$ or $dX \in (X)$. These derivations do form a subset but different derivations may correspond to the same "quotient"-derivation on $\mathbb{C}[X]/(X^2)$.

17. Put $A = A_1 \times A_2$ and d a derivation of A. Then $d(a, o) = d'((1, o)(a, o)) = d(1, o) + (1, o)d(a, o) \in A_1 \times o$, thus $d|A_1 \times o$ is a derivation of A_1 and similar for $d|o \times A_2$. Thus $d = d|A_1 \times o \oplus d|0A_2$.

18. *Wrong.* $\mathbb{C}[X^2] \subset \mathbb{C}[X]$ and $\mathbb{C}[X^2] \cong \mathbb{C}[X]$, hence $\partial : f(X^2) \to f'(X^2)$ is a derivation of $\mathbb{C}[X^2]$. This derivation is not of the form $p(X)\frac{d}{dX}$ because then $\partial X^2 = 1 = p(X)2X$, a contradiction.

19. *Wrong.* The derivations of $M_2(\mathbb{C})$ are inner but E_{12} is a zerodivisor.

20. *Right.* Looking at $\mathbb{C}[X]$ as a subalgebra of $\mathbb{A}_1(\mathbb{C})$ we have $[f(x)y, x] = f(x)$ and then $[f(x)y, -]|\mathbb{C}[X]$ is $f(x).\partial_x$.

21. Too easy.

22. *Wrong.* $\mathrm{Inn}_K K[X] = 0$ but this is not $K[X]$ with the zero bracket.

23. *Wrong* Let the Lie algebra have basic vectors v_1 and v_2 and assume $[v_1, v_2] = av_1 + bv_2$. If the Lie algebra is not abelian we may assume that $a \neq 0$ (up to interchanging the basis vectors. Form a new basis, $e_1 = v_i + \frac{b}{a}v_2, e_2 = \frac{v_2}{a}$, then $[e_1, e_2] = e_1$. Now put $V = K(se_1 + te_2)$ and assume it is a Lie ideal, then $[se_1 + te_2, c_2] = se_1 \in V, i.e. t = 0$, and also $[se_1 + te_2, e_1] = -te_1 \in V$ or $s = 0$. Hence V is trivial.

24. *Right.* The algebra $\mathbb{C}[X]/I$ is itself finite dimensional.

25. *Wrong.* The usual derivative $\frac{d}{dX}$ is not trivial on $K(X)$.

26. *Right.* Because $[\delta_x, \delta] = -\delta_{\delta x}$ is inner.

27. *Wrong.* Put $A = K[X]/(X^2)$ with basis 1 and $\overline{X} = X\mathrm{mod}(X^2)$. Then $\mathrm{Der}_K A$ may be identified with the K-linear maps having $\overline{X}$ as an eigenvector and mapping 1 to zero. This set is closed under composition.

28. *Right.* The commutator $[x, x] = 0$ for every element.

29. *Right.* The map $[a, -] : g \to g$ is linear and g is finite dimensional over $\mathbb{C}$, hence $[a, -]$ has an eigenvector, say $b \neq 0$. One easily $[b, [a, b]] = [b, \lambda] = 0$, some $\lambda \in \mathbb{C}$.

30. *Right.* Let δ be an $\mathbb{R}$-derivation of $\mathbb{H}$, say :

$$\delta(i) = a_0 + a_1 i + a_2 j + a_3 k,$$
$$\delta(j) = b_0 + b_1 i + b_2 j + b_3 k,$$

Then $\delta(i^2) = 0 = i\delta(i) + \delta(i)i = -2a_1 + 2a_0 i$, thus $a_1 = a_0 = 0$, similarly $b_0 = b_2 = 0$. From $\delta(ij) = -\delta(ji)$ we calculate :

$$i\delta(j) + \delta(i)j = -a_2 - b_1 - b_3 j + a_3 i = -\delta(ji) = a_2 + b_1 - b_3 j + a_3 i$$

hence $a_2 = -b_1$. Then $\delta = \frac{1}{2}[a_2 k + a_3 j + b_3 i, -]$, hence every derivation of $\mathbb{H}$ over $\mathbb{R}$ is inner. Thus we have $\mathrm{Der}_{\mathbb{R}}\mathbb{H}/ = \mathbb{H}/\mathbb{R} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ and $[\frac{1}{2}i, \frac{1}{2}j] = \frac{1}{2}k, [\frac{1}{2}j, \frac{1}{2}k] = \frac{1}{2}i, [\frac{1}{2}k, \frac{1}{2}i] = \frac{1}{2}j$ and thus ϕ: $\mathrm{Der}_{\mathbb{R}}\mathbb{H} \to so_3(\mathbb{R}) = \mathbb{R}^3, \times$, defined by $a_1 i + a_2 j + a_3 k \mapsto 2(a_1, a_2, a_3)$ is an isomorphism, where $\mathbb{R}^3\times$ is the 3-dimensional $\mathbb{R}$-space with the vectorial product $\times$.

31. *Wrong.* Look at $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}[T]$. The elements iT and jT generate an infinite dimensional Lie algebra with the commutator bracket i.e.

$$\mathbb{R}\{iT, jT, kT^2, iT^3, jT^3, \ldots\}$$

32. *Wrong.* The dimension of $sl_n(\mathbb{C})$ is $n^2 - 1$ (all matrices of trace zero) and $l^2 - 1 + k^2 - 1 \neq (l + k)^2 - 1$.

33. *Right.* Choose as a new basis $\{\lambda e_i, i\}$, the new structure constants are $\lambda c^k_{j_j}$ now.

34. *Wrong.* The groups $O_3(\mathbb{R})$ and $SO_3(\mathbb{R})$ have the same Lie algebras.

35. *Wrong.* The subalgebra generated by the matrix with a 1 only in the upper right hand corner is a Lie ideal.

36. *Wrong.* The algebra $\mathbb{C}$ with the zerobracket is simple. If $g \neq \mathbb{C}$ then the statement is right in view of the main theorem for simple Lie algebras (every two generators E_i, F_i generate an sl_2-subalgebra).

37. *Wrong.* Every simple Lie algebra does not have Lie ideals hence it can only be mapped surjectively to itself.

38. *Wrong.* The enveloping algebra of $\mathbb{C}$ with the zero bracket is $\mathbb{C}[X]$.

39. *Wrong.* $\mathbb{C}^{\oplus n}$ only has the zero derivation (the zero Lie algebra is not assumed to be simple).

40. *Wrong.* Such path-algebra is finite dimensional and thus it cannot be the enveloping algebra of a nonzero Lie algebra.

41. *Wrong.* The algebra $\mathbb{C} \subset M_n(\mathbb{C})$ with the zero bracket is simple. If $g \neq \mathbb{C}$ then the statement is right because the subspace of matrices with trace zero is a Lie ideal. If dim $g > 1$ then the latter subspace is not trivial because it is the solution space of a linear equation, hence g must consist completely of true zero matrices.

42. *Right.* Let $\dim_{\mathbb{C}} g \geq 2$ and take X, Y linearly independent in g. Then $\mathbb{C}X$ and $\mathbb{C}Y$ are Lie ideals, thus $[X, Y] \in \mathbb{C}X \cap \mathbb{C}Y$ yields $[X, Y] = 0$.

43. *Wrong.* If we consider $\mathbb{C}X \otimes \mathbb{C}Y$ with $[X, Y] = Y$ then we obtain a counter-example.

44. *Wrong.* The example in 4.3. is not simple and it cannot be written as the direct sum of two simple Lie algebras (because otherwise it would be the direct sum of two one dimensional Lie algebras and as such it would have to be Abelian).

45. *Wrong.* Counterexample : $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

46. *Wrong.* We have $\mathrm{sl}_n(\mathbb{C}) \subset M_n(\mathbb{C})$ of dimension $n^2 - 1$.

47. *Right.* Every finite dimensional Lie algebra can be embedded in a matrix algebra. Extend this embedding to the level of enveloping algebras.

48. *Right.* The universal enveloping algebra is not finite dimensional.

3.4 Right or Wrong

1. *Wrong.* $\mathbb{C}[X,Y]/(X^2,Y^2,XY)$ is $\mathbb{Z}/2\mathbb{Z}$-graded but it has $\mathbb{C}$-dimension 3.

2. *Right.* Look at a G-graded algebra and select a set of homogeneous generators, $\mathcal{X}$ say. Look at the free algebra $K < \mathcal{X} >$ and give every element of $\mathcal{X}$ then corresponding G-degree. This defines a G-gradation on the free algebra and the canonical map $T : K < \mathcal{X} > \to A$ is graded of degree $e \in G$.

3. *Wrong.* If $\xi \in K[X]_{\sigma_1}, \eta \in K[X]_{\sigma_2}$, then $\xi\eta \in K[X]_{\sigma_1\sigma_2}$ but since $\xi\eta = \eta\xi$ also $\xi\eta \in K[X]_{\sigma_2\sigma_1}$, leading either to $\xi\eta = 0$ what is impossible or $\sigma_1\sigma_1 = \sigma_1\sigma_2$. We can choose σ_1 and σ_2 such that $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$ by the noncommutativity of $G = S_3$.

4. *Wrong.* We have an ideal $2\mathbb{Z}$ in $\mathbb{Z}$ but $K[X^2] = K[X]_{2\mathbb{Z}}$ is not an ideal.

5. *Wrong.* Look at the $\mathbb{Z}$-graded ring $K[X, X^{-1}]$ with $\deg X = 1$, $I = \oplus_{i\neq 0} KX^i$ is not an ideal as it contains X but not $X^{-1}X = 1$.

6. *Right.* Indeed $[x_i, x_j]$ has degree two, since $[x_i, x_j] = \sum_k c_{ij}^k x_k$ it must be of degree one, hence $[x_i, x_j] = 0$ for all i, j or g is Abelian.

7. *Right.* Easy enough.

8. *Right.* Put $a \in R_h$ and decompose a^{-1} into $\sum_g x_g$. Then $1 = \sum_g x_g a$, hence $1 = x_{h^{-1}}a$, from $1 = \sum_g a x_g$ it follows $1 = a x_{h^{-1}}$ hence $x_{h^{-1}} = a^{-1}$.

9. *Wrong.* $\mathbb{C}(X) \neq \oplus_k \mathbb{C}X^k$.

10. *Right.* Choose a basis for every homogeneous component and take the union of these.

11. *Wrong.* $\mathbb{C}[X]/(X^2)$ is $\mathbb{Z}$-graded with $\deg\overline{X} = 1$, $\overline{X} = X \bmod (X^2)$, but $1, 1 + \overline{X}$ is a $\mathbb{C}$-basis.

12. *Right.* $A \oplus B = (\oplus_i A_i) \oplus (\oplus_j B_j) = \oplus_i (A_i \oplus B_i)$ and $(A_i \oplus B_i)(A_j \oplus B_j) = (A_iA_j \oplus B_iB_j \subset A_{i+j} \oplus B_{i+j} = (A \oplus B)_{i+j}$.

13. *Wrong.* Look at $\mathbb{C}[X,Y]/(X^2,Y^2,XY)$ graded by putting $\deg\overline{X} = 1$, $\deg\overline{Y} = 2$; then $(X+Y, X-Y)$ is a graded ideal (equal to (X,Y)).

14. *Wrong.* Give $\mathbb{C}$ a $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots (= \mathbb{Z}_2^{(\infty)})$-gradation by putting $\deg(i) = (1,1,\ldots)$.

15. *Right.* $M_n(K)$ is generated by elements E_{ij} with relations : $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Split $\{1,\ldots,n\}$ in two disjoint subsets. Write $i \sim j$ if they are in the same subset. Put $\deg E_{ij} = 0$ if $i \sim j$ and 1 otherwise. This is a gradation because if $E_{ij}E_{kl} = E_{jl} \neq 0$, then $j = k$ and then $j \sim l \Leftrightarrow (i \sim j$ and $k \sim l)$ or $(i \not\sim j,\ k \not\sim l)$.

16. *Right.* Let R be a non-trivially Z-graded finite dimensional algebra and consider $x \in R$ homogeneous of degree $m \neq 0$. Then $x^n \in R_{nm}$. Since only a finite number of homogeneous components of R can be nonzero we have $x^n = 0$ for n large enough.

17. *Right.* The path-algebra is the quotient of the free algebra (with generators the arrows a_i and vertices v_j). Give vertices degree zero and arrows any arbitrary degree. The defining ideal contains relations of the form : $a_ia_j, a_iv_j(-a_i), v_ja_i(-a_i), v_i^2 - v_i$. All these relations are now homogeneous, hence the ideal is graded in the free algebra hence the path algebra inherites this gradation.

18. *Right.* Because $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (n-terms) has no nilpotents so we may apply question 16. above.

19. *Wrong.* Look again at $\mathbb{C}[X,Y]/(X^2,Y^2,XY)$. We may choose infinitely many bases of the form $1, X, aX + bY$. Give everything degree zero except $\deg(aX + bY) = 1$. This defines a gradation and for different a and b the gradations are different because $\mathbb{C}(aX + bY) \neq \mathbb{C}(cX + dY)$ if $(a,b) \neq (c,d)$.

20. *Wrong.* We may define an S_3-gradation on : $K[X_1,\ldots,X_5]/(X_iX_j, 1 \le i,j \le 5)$ by giving every generator $\overline{X}_1,\ldots,\overline{X}_5$ a different degree, different from $e \in \mathcal{S}_3$. This algebra is commutative and cannot be mapped surjectively onto $K[\mathcal{S}_3]$.

21. *Right.* The relations defining the exterior algebra in the free algebra, i.e. $X_iX_j + X_jX_i$ are homogeneous if we put $\deg X_i = 1$ for all i.

22. *Wrong.* Define a gradation on $M_2(\mathbb{C})$ by putting $\deg E_{12} = 1$, $\deg E_{21} = -1$. Then $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is idempotent but not homogeneous.

23. *Right.* If $\lambda \in K$ has a positive degree $k \in \mathbb{N}$ then $\lambda^{-1} = f_0 + f_1 + \ldots + f_n$, $\lambda(f_0 + \ldots + f_n) = 1$ has degree zero but every λf_i has degree strictly larger than zero, hence $\lambda f_i = 0$ for all $-i$, contradiction.

24. *Wrong.* Look at the $\mathbb{Z}$-graded $K[X, X^{-1}]$ with $\deg.X = 1$.

25. *Right.* Assume $\{a\}$ is closed in the filtration topology. The map $g : x \mapsto x + a - f$ is continuous, thus $g^{-1} = \{f\}$ is the inverse image of a closed set, thus also closed.

26. *Right.* Because the standard filtration on $K[X]$ is discrete $(F_{-1}, K[X] = 0)$.

27. *Wrong.* $\mathbb{Q}_2 = 0 + (2) \cup 1 + (2)$ is the union of two open sets.

28. Assume that A has infinitely many elements and that it is compact for the filtration topology. As $|A| = \infty$ this topology is not discrete, hence $F_iA \neq 0$ for all $i \in \mathbb{Z}$. We have $F_nA = A$ for some n because otherwise we have an infinite cover $\{F_nA, n \in \mathbb{Z}\}$ not having a finite subcover. Now look at $\{b + F_iA, b \in F_{i+1}A - F_iA\}$. Two such $b + F_iA$ are either equal or disjunct. Therefore this infinite cover cannot have a finite subcover.

29. *Right.* Suppose FQ is such that $\widehat{Q} = \mathbb{R}$. Taking then an open U in $\mathbb{R}$ we must have $U \cap Q$ open in the filtration topology of Q. hence $U \cap Q$ must contain some $x + F_i\mathbb{Q}$. If $a \in F_iQ$ then also $na \in F_iQ$ and thus F_iQ is not finite, hence U is unlimited, a contradiction.

30. *Right.* If $\sum a_iz^i = \sum b_iz^i$ with $a_n = 1 \neq b_n = 0$ being the first different digit. Then we have : $\sum a_iz^i - \sum b_iz^i = 2^n + 2^{n+1}a$. For every $x \in R$, $2^n + 2^{n+1}x \notin (2)^{n+1}$, thus $0 + (2)^{n+1} \in \sum a_iz^i - \sum b_iz^i$. Two formal sums are therefore equal if and only if all digits are equal. Every number also can be written as such a formal sum, i.e. take $a \in \mathbb{Z}_{(2)}$ then we may consider a as a limit of a sequence of integers $(z_0, zi_1, \ldots)$. Adding to each z_i an element of (z^i) does not change the limit. This means we may choose all z_i positive. Every positive integer has such a decomposition, take the i^{th} digit of a to be the limit of the i^{th} digit of the z_j, this yields a power series description for a.

    - *Example 1.* Put $a = -1$, then $(-1, -1, \ldots)$ is a sequence converging to -1, so also $(0, 1, 3, 7, \ldots)$ and the power series for -1 is $1 + 2 + 4 + 8 + \ldots$.
    - *Example 2.* Put $a = 1/3$. Then $(1, 3, 11, \frac{129}{3} = 43, \frac{513}{3} = 171, \ldots)$ is a sequence converging to $1/3$, therefore the power series for $\frac{1}{3}$ is $1 + 2 + 8 + 128 + \ldots$

31. *Right.* Because $(X^{n-1})/(X^n) \cong \mathbb{C}X^{n-1}$ and hence $G_X(\mathbb{C}[X]) = \ldots \oplus \mathbb{C}X^2 \oplus \mathbb{C}X \oplus \mathbb{C} \oplus 0 \oplus 0 \oplus \ldots$. The multiplication is as expected but the gradation is now negative.

32. *Wrong.* Take $R = K[X, Y]/(X^2)$ with the trivial filtration $F_{-i}R = (Y^i)$ for $i > 0$ and $F_0R = R$. The set of zero divisors of R is the ideal $(\overline{X})$. To be open the latter must contain some $\overline{X} + F_iR$ but this is impossible because $\overline{X} + Y^i$ cannot be a zero divisor.

33. *Wrong.* Again take $R = K[X, Y]/(X^2)$ with $F_{-i}R = (Y^i)$ for $i > 0$ and $F_0R = R$. If the ideal of zerodivisors $(\overline{X})$ (as in question 32. has to be closed then (observe that 0 is not considered as a zero divisor) some $0 + F_iR$ must be disjoint from $(\overline{X})$, but this is impossible because $0 + \overline{X}Y^i$ is always a zero divisor.

34. *Right.* Since every ideal of A is finitely generated, $\mathcal{N} = Aa_1 + \ldots + Aa_k$. If $a_i^N = 0$ for $i \in \{1, \ldots, k\}$ then we have $\mathcal{N}^{kN} = 0$ because :

$$(r_1a_1 + \ldots + r_ka_k)^{Nk} = \sum_{i_1+\ldots+i_k=Nk} (r_1a_1)^{i_1} \ldots (r_ka_k)^{i_k}$$

    and at least one i_j is larger than N. The right-hand member is therefore zero. Hence $F_{-kN}R = 0$ and the topology is discrete.

35. Consider $A = M_n(K)$. For every $d_1, \ldots, d_n \in \mathbb{Z}$ consider $M_n(K)_\lambda = (K_{\lambda+d_i-d_j})_{ij}$; this defines a $\mathbb{Z}$-gradation on $M_n(K)$.

36. *Wrong.* For any Lie algebra of dimension n over K, g say, the associated graded of $U_K(g)$ is always the polynomial ring $K[X_1, \ldots, X_n]$, hence two non-isomorphisc n-dimensional Lie algebras yield the same associated graded algebra.

37. *Right.* Since $G(U_K(g))$ is a domain, the principal symbol map σ is multiplicative hence $U_K(g)$ is a domain too.

38. *Right.* If $a^{-1} \in U_K(g) = A$ then $a^{-1} \in F_nA - F_{n-1}A$ for some $n \in \mathbb{N}$ hence $\sigma(a^{-1}) \in G(A)_n$. Sine $\sigma(a)\sigma(a^{-1}) \neq 0$ it is $\sigma(aa^{-1}) = \sigma(1)$ or $\sigma(a)\sigma(s^{-1}) = 1$ or $\deg\sigma(a) = \deg\sigma(a^{-1}) = 0$ since $G(A)$ is positively graded, i.e. $a^{-1} \in F_0A = K$, $a \in K$.

39. *Right.* Obvious from definition of direct sum.

40. Suppose, $F_kR \supsetneq F_0R$ for $k > 0$. Pick $x \in F_kR - F_0R$ then $\sigma(x+a) = \sigma(x)$ yields $\sigma(a) = 0$ for every $a \in F_0R$, in fact for every $a \in F_lR$, $l \leq 0$. Consequently $F_0R = F_{-1}R = F_lR + \ldots$ for $l \leq 0$, contradicting the separatedness.

41. *Right.* It is easily verified that $F'_kRF'_lR \subset F'_{l+k}R$ and $1 \in F'_0R$. The filtration $F'R$ is the step-wise version of FR with step of lentgh two !

42. *Right.* Both define the discrete topology.

43. *Wrong.* The generator-filtration topology is discrete and hence every subset is open, then if the statement were true every subset of $\mathbb{C}^\eta$ is open, contradiction.

44. *Right.* The I-adic filtration defines a Hausdorff topology if and only if $\cap_n I^n = 0$. If I contains a nonzero idempotent e then $e \in \cap_n I^n$.

45. *Right.* We have $x + (I^n) = \cup_{y \in I^n} x + y + (I^{2n})$ is a union of open sets in the (I^2)-adic topology, hence open in this topology too.

46. *Wrong.* Filter $\mathbb{C}$ by $0 \subset \ldots \subset 0 \subset \mathbb{C} \subset \mathbb{C} \subset \ldots \subset \mathbb{C} \subset$; then $\widetilde{\mathbb{C}} \cong \mathbb{C}[T]$.

47. *Right.* We have $fT^n \mapsto \lambda^n fT^n$, this map has an inverse $fT^n \to \lambda^{-n} fT^n$.

48. *Wrong.* The Rees ring is graded so it has the grading filtration $F_i\widetilde{R} = \oplus_{j \leq i} \widetilde{R}_j$. Now look at $\mathbb{C}$ with the filtration defined in 46., then $\widetilde{\mathbb{C}} = \mathbb{C}[T]$ and the filtration on this is the standard filtration. Hence $(\widetilde{\mathbb{C}})^\sim = \mathbb{C}[U, TU] \cong \mathbb{C}[X, Y]$ and this ring is not isomorphic to $\mathbb{C}[X]$.

49. *Wrong.* $\mathbb{C} = \mathbb{C}[X]$ as in 46.

50. *Wrong.* Same counterexample as 49.

4.3.15.

1. Follows from 2. if we check that $M_n(\Delta)$ is Artinian. Since Δ is a skewfield we may use the left dimension, and $\mathrm{ldim}_\Delta M_n(\Delta) = n^2 < \infty$, so ldim_Δ strictly descends on left ideals $L_i \subset L_{i-1}$ in $M_n(\Delta)$. Similarly for right ideals using the right dimension.

2. Follow the hint.

3. Since every ideal is generated by an idempotent, the hint yields the proof.

References

1. V. Bavula, F. Van Oystaeyen, *Simple Holonomic Modules over the Second Weyl Algebra*, Adv. in Math. Vol 150, No. 1, 2000, 80-116.

2. J. E. Björk, *Rings of Differential Operators*, North Holland.

3. W. Bruns, U. Vetter, *Determinantal Rings*, Lect. Notes in Math. 1327, Springer Verlag, Berlin, 1988.

4. J. Cozzens, C. Faith, *Simple Noetherian Rings*, Cambridge Univ. Press, Cambridge, 1975.

5. J. Dixmier, *Algèbres Enveloppantes*, Gauthier-Villars, 1974 (reedition in 1998).

6. W. Fulton, J. Harris, *Representation Theory. A First Course*, Grad. Text in Math., 129, Springer Verlag, Berlin.

7. A. Goldie, *The Structure of Noetherian Rings*, Lect. Notes in Math. 246, Springer Verlag, Berlin, 1972.

8. M. Hall, *The Theory of Groups*, Amer. Math. Soc., Chelsea Publ. Co, 1959.

9. I. Herstein, *Noncommutative Rings*, Carus Math. Monographs, Math. Assoc. of Am., New edition in 2005 by Cambridge Univ. Press.

10. J. Humphriys, *Introduction to Lie Algebras and Representation Theory*, Grad. Text in Math. 9, Springer Verlag, Berlin, 1972 (seventh printing, 1997).

11. N. Jacobson, *Structure of Rings*, Amer. Math. Soc., 1984.

12. L. Le Bruyn, M. van den Bergh, F. Van Oystaeyen, *Graded Orders*, Birkhauser, Boston, 1988.

13. H. Li, F. van Oystaeyen, *Zariskian Filtration*, K-monographs in Math., Vol. 2, Kluwer Acad. Publ., Dordrecht, 1996.

14. J. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, Wiley, Interscience, New York, 1987.

15. S. McLane, *Categories for the Working Mathematician*, Grad. Text in Math. Vol. 5, Springer Verlag, Berlin, 1998.

16. C. Năstăsescu, F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, Lect. Notes in Math., 758, Springer Verlag, Berlin, 1979.

17. C. Năstăsescu, F. Van Oystaeyen, *Graded Ring Theory*, North Holland, 1982, Dordrecht.

18. C. Năstăsescu, F. Van Oystaeyen, *Dimensions of Ring Theory*, Math. and its Applications, Vol. 36, D. Reidel Publ. Cie, Dordrecht, 1987.

19. D. G. Northcott, *An Introduction to Homological Algebra*, Cambridge Univ. Press, London, 1960.

20. F. Van Oystaeyen, *Algebraic Geometry for Associative Algebras*, Monographs in Pure and Applied Math. Vol. 227, M. Dekker, New York, 2000.